Superconducting devices, junctions and application

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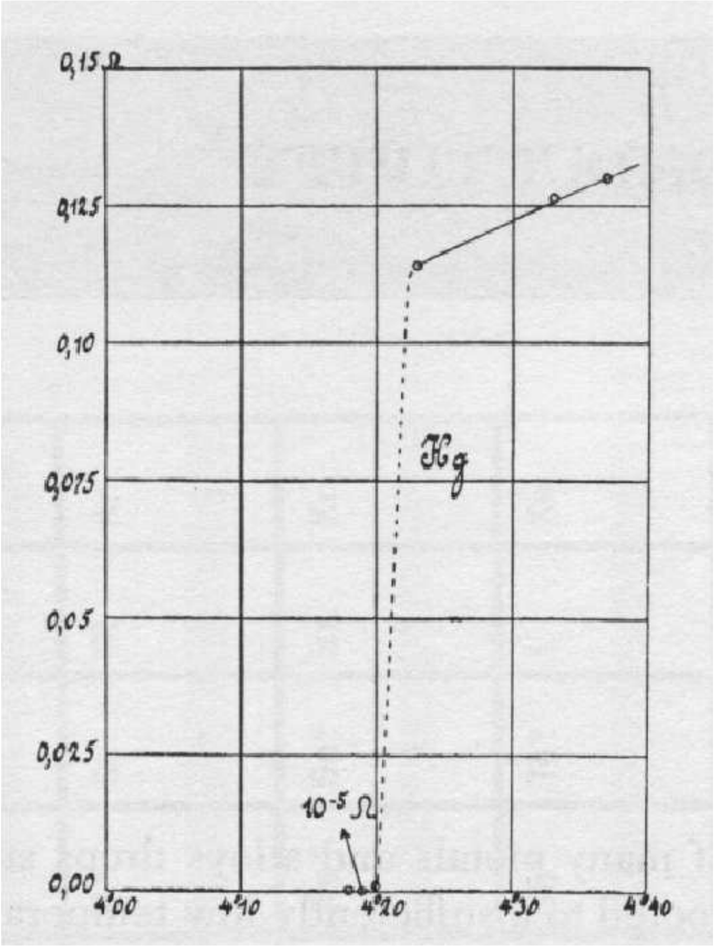
TIGP Advanced Nano

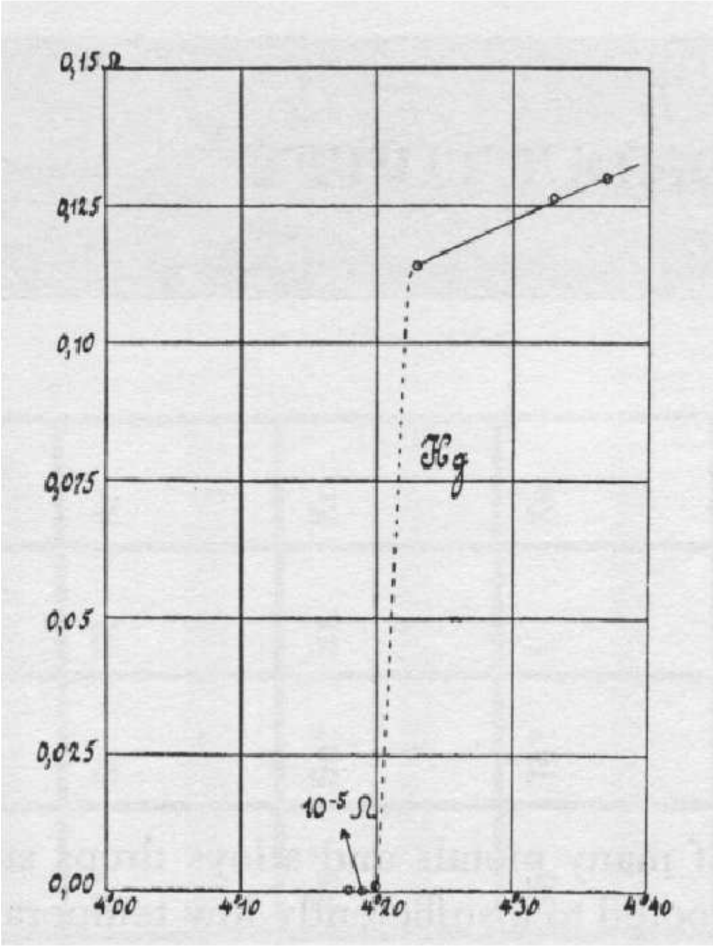
Superconductor

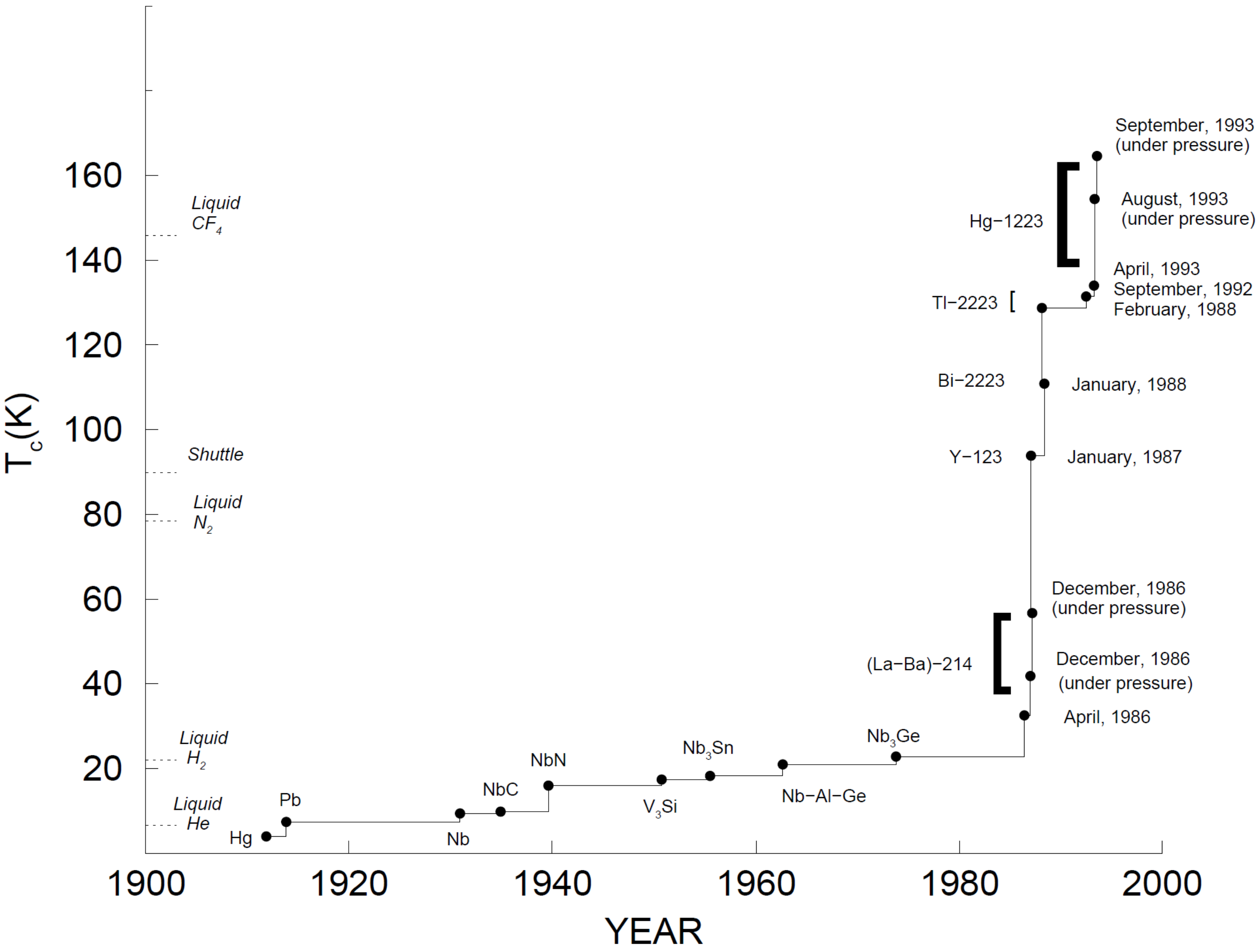


Kittel, Introduction
to Solid State Physics

1911 by Kamerlingh Onnes

Superconductor





Kittel, Introduction
to Solid State Physics

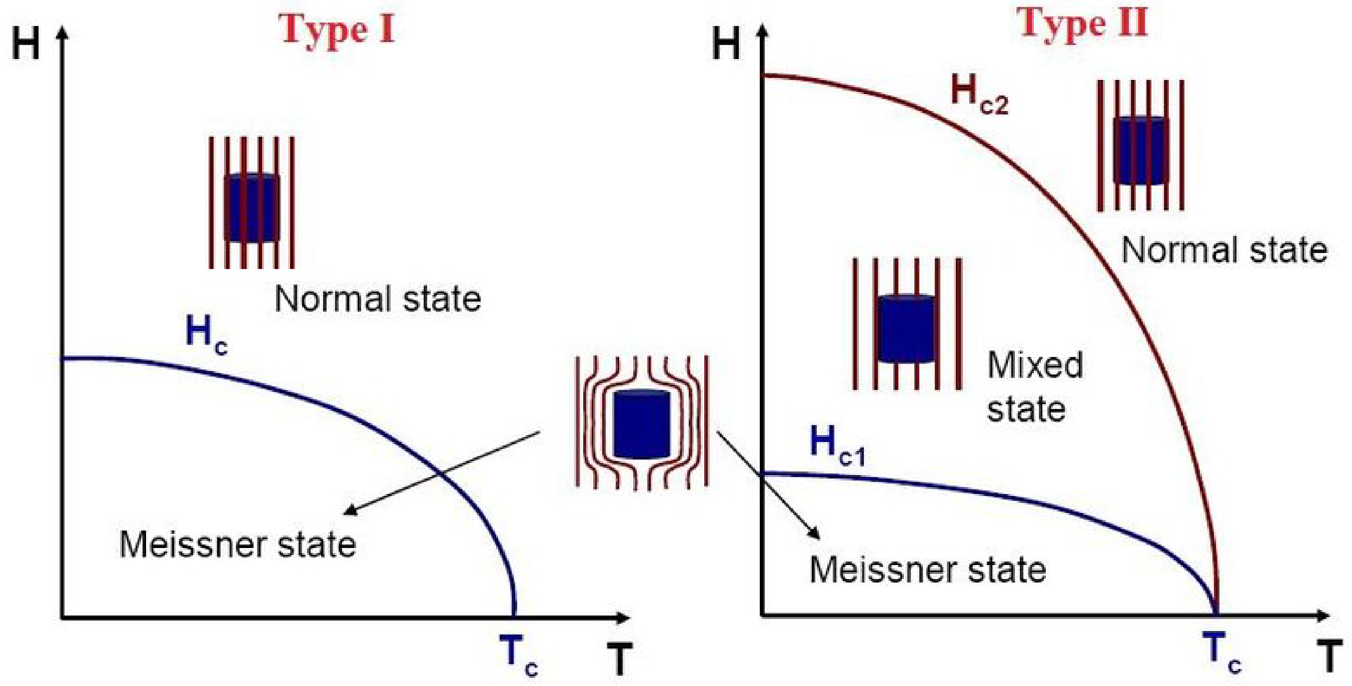
Superconductor

Phenomena SC Properties:

1. Zero resistance: No resistance is detectable, even for high scattering rates of conduction electrons.
2. Absence of thermoelectric effects: No Seebeck voltage, no Peltier heat, no Thomson heat is detectable
3. Ideal diamagnetism: $\chi_m = -1$. Weak magnetic fields are completely screened away from the bulk of a superconductor
4. Meissner effect: If a superconductor is cooled down in the presence of a weak magnetic field, below Tc, the field is completely expelled from the bulk of the superconductor.
5. Flux quantization: The magnetic flux through a superconducting ring is quantized and constant in time. This phenomenon was theoretically predicted by F. London in 1950 and experimentally verified 1961.

Helmut Eschrig, THEORY OF SUPERCONDUCTIVITY

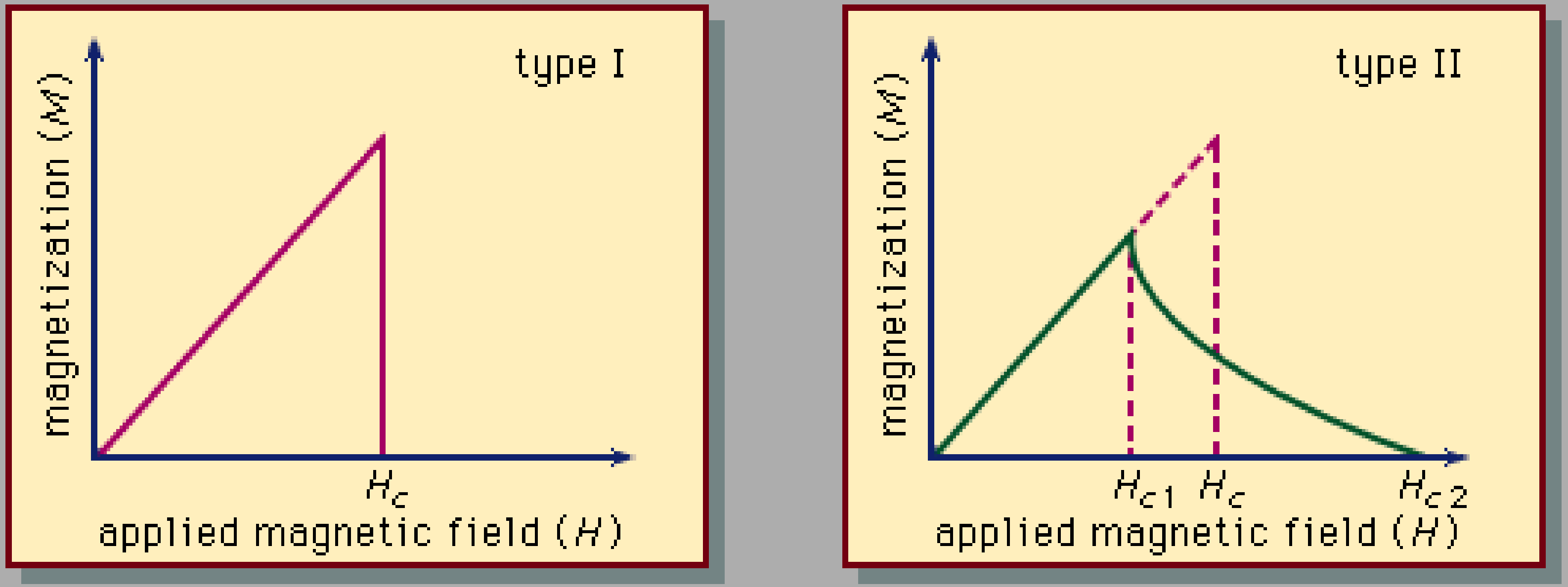
Superconductor



PABLO CAYADO LLOSA thesis

	T_c, K	H_c, Oe	H_{c2}, Oe	λ_L, Å	ξ_0, Å	κ	Type
Al	1.18	105		500	16000	0.01	I
Hg	4.15	400		400			I
Nb	9.25	1600	2700	470	390	1.2	II
Pb	7.2	800		390	830	0.47	I
Sn	3.7	305		510	2300	0.15	I
In	3.4	300		400	3000		I
V	5.3	1020		400	~300	~ 0.7	II

Superconductor



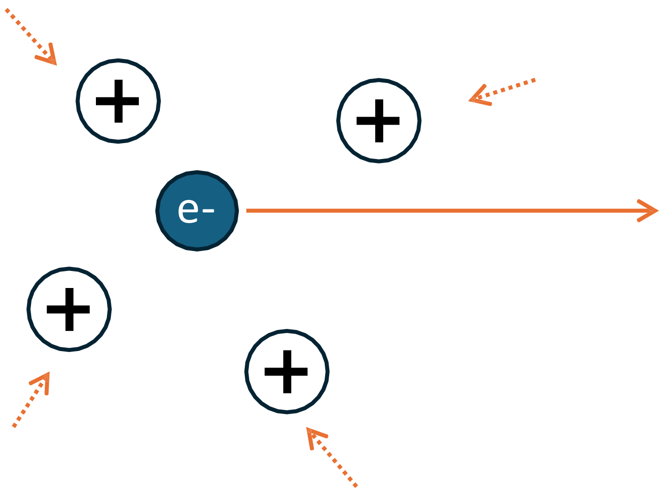
For a type I superconductor, magnetic flux is expelled, producing a magnetization (M) that increases with magnetic field (H) until a critical field (H_c) is reached, at which it falls to zero as with a normal conductor.

A type II superconductor has two critical magnetic fields (H_{c1} and H_{c2}); below H_{c1} type II behaves as type I, and above H_{c2} it becomes normal.

Superconductor

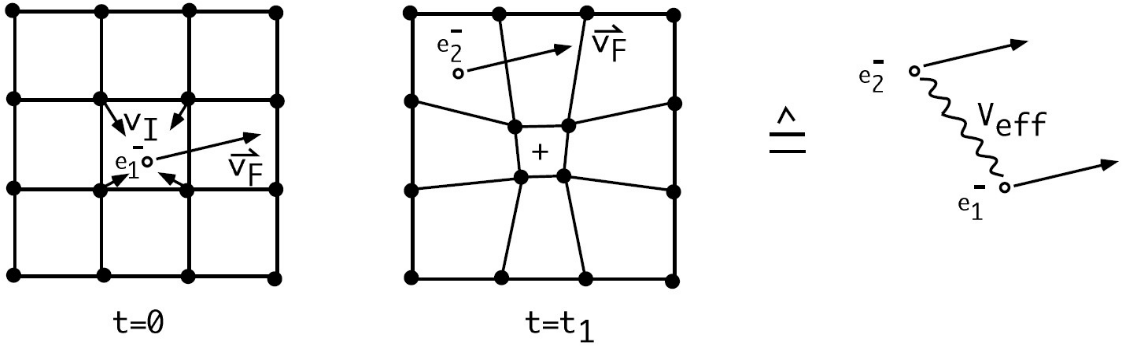
Frölich 1950

Provides an idea of why there is an attractive potential between electrons.



Superconductor

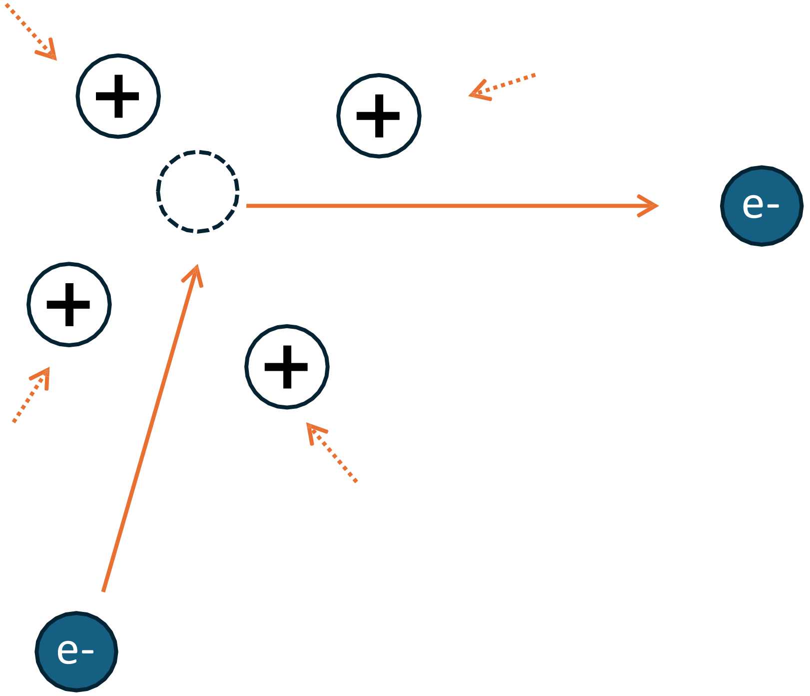
The lattice effect can be attributed to phonons.
The electron-phonon couple provides an additional coupling potential



Frölich 1950

Provides an idea of why there is an attractive potential between electrons.

$$V(\mathbf{q},\omega) = \frac{4\pi e^2}{q^2+k_s^2} + \frac{4\pi e^2}{q^2+k_s^2}\,\frac{\omega_q^2}{\omega^2-\omega_q^2},$$



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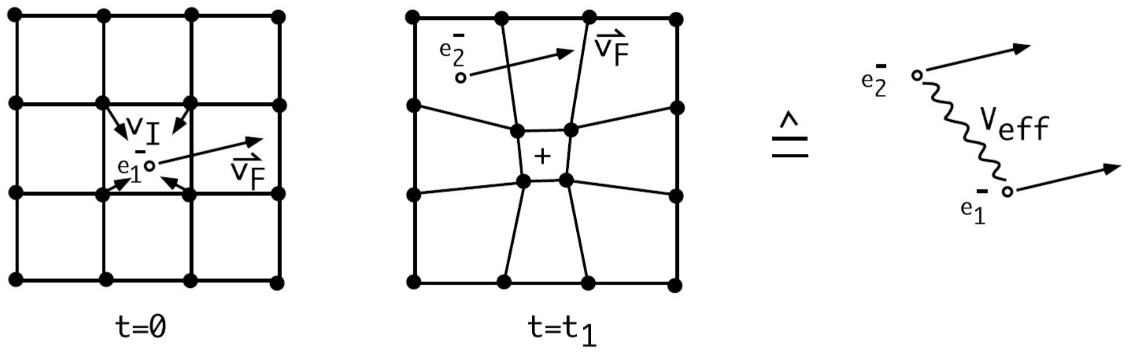

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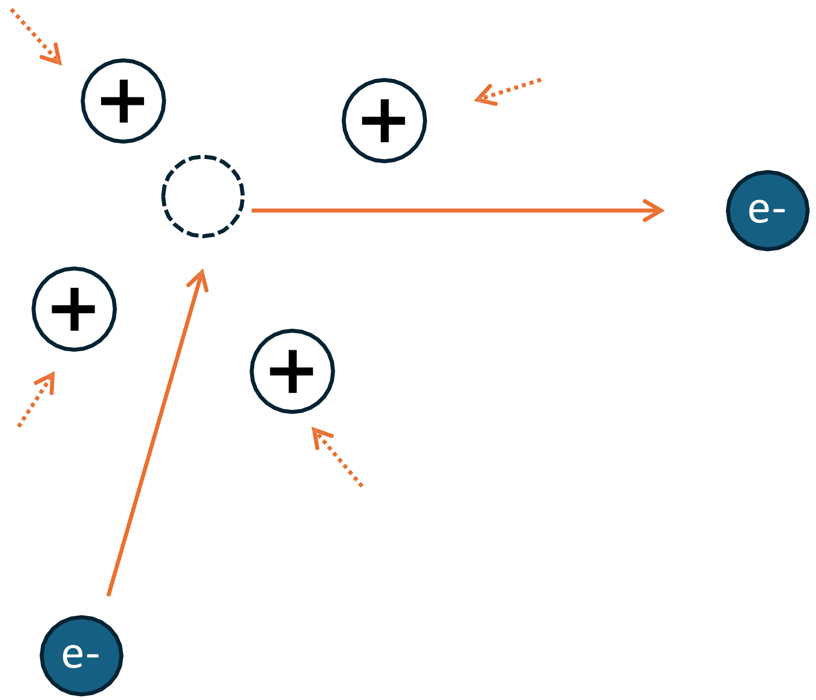

$$V(\mathbf{q},\omega)=\frac{4\pi e^2}{q^2+k_s^2}+\frac{4\pi e^2}{q^2+k_s^2}\frac{\omega_q^2}{\omega^2-\omega_q^2},$$

Attractive term for $\omega^2 < \omega_q^2$

A few issues, V($\omega \to 0$)=0, for low frequency, always attractive

Superconductor

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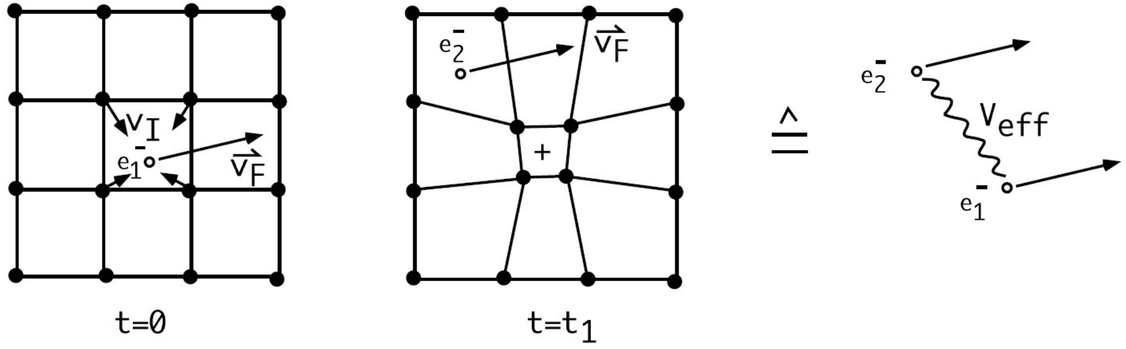

<u>Frölich 1950</u>
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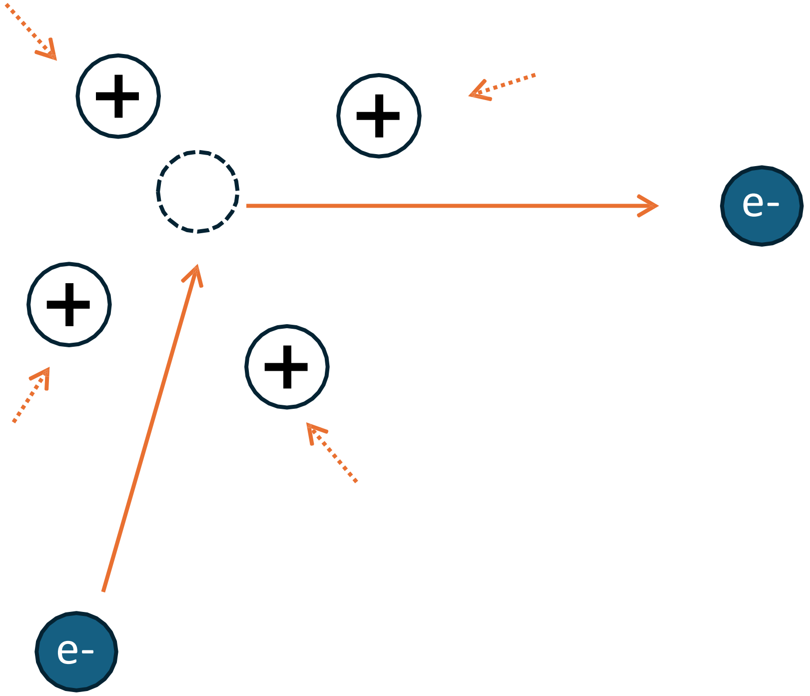


$$V(\mathbf{q},\omega) = \frac{4\pi e^2}{q^2+k_s^2} + \frac{4\pi e^2}{q^2+k_s^2}\,\frac{\omega_q^2}{\omega^2-\omega_q^2},$$

<u>Leon Cooper</u>
Provides to pairing wave function between two fermionic electrons.

$$\psi(r_1,r_2) = \sum_{kq} u_k(q) e^{ik\cdot r_1} e^{-i(k+q)\cdot r_2}\zeta$$

$$\Rightarrow \quad \Delta_{\text{Cooper}} \equiv 2\epsilon_F - E \simeq 2\omega_c e^{-2/N_0 V}.$$

Superconductor

London theory:

At T=0, we expect to have the supercurrent as a property of quantum states

$n_B = |\Psi|^2,$ n_B is the bosonic density, and Ψ is the corresponding field amplitude

From Schrödinger's equation:

$$\frac{1}{2m_B}\left(\frac{\hbar}{i}\frac{\partial}{\partial \boldsymbol{r}} - q\boldsymbol{A}\right)^2 \Psi + qU\Psi = (E - \mu_B)\Psi,$$

$$\Psi(\boldsymbol{r}, t) = \sqrt{n_B} e^{i\theta(\boldsymbol{r}, t)},$$

Skipping the immediate steps, we arrived

$$\Lambda \boldsymbol{j}_s = \frac{\hbar}{q}\frac{\partial \theta}{\partial \boldsymbol{r}} - \boldsymbol{A}, \quad \Lambda = \frac{m_B}{n_B q^2}.$$

Superconduc

London theory:

At T=0, we expect to ł

$$n_B = |\Psi|^2, \qquad \text{r}$$

From Schrödinger's e

$$\frac{1}{2m_B}\left(\frac{\hbar}{i}\frac{\partial}{\partial \boldsymbol{r}}\right.$$

Skipping the immedia

London Theory

The London equations are a set of phenomenological equations in an attempt to describe the Meissner effect.

Newton's law (inertial response) for applied electric field

$$F = m\frac{d}{dt}(v_s) \Rightarrow eE = m\frac{d}{dt}\left(\frac{J_S}{n_s e}\right) \Rightarrow \boxed{\frac{n_s e^2 E}{m} = \frac{dJ_S}{dt}}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

Maxwell Eqs

Here supercurrent density is $J_s = n_s e v_s$

$$\vec{\nabla} \times \frac{n_s e^2 \vec{E}}{m} = \vec{\nabla} \times \frac{d\vec{J}_S}{dt} \Rightarrow -\frac{n_s e^2}{m}\frac{d\vec{B}}{dt} = \vec{\nabla} \times \frac{d\vec{J}_S}{dt}$$

$$\Rightarrow \frac{d}{dt}\left[\vec{\nabla} \times \vec{J}_S + \frac{n_s e^2}{m}\vec{B}\right] = 0 \Rightarrow \boxed{\vec{\nabla} \times \vec{J}_S = -\frac{n_s e^2}{m}\vec{B}}$$

We know B = 0 inside superconductors

Superconductor

London theory:

The first London equation:

$$\frac{\partial(\Lambda \boldsymbol{j}_s)}{\partial t} = \boldsymbol{E}_{\text{eff}}$$

"A supercurrent is freely accelerated by an applied voltage, or, in a bulk superconductor with no supercurrent or with a stationary supercurrent there is no effective electric field (constant electrochemical potential)."

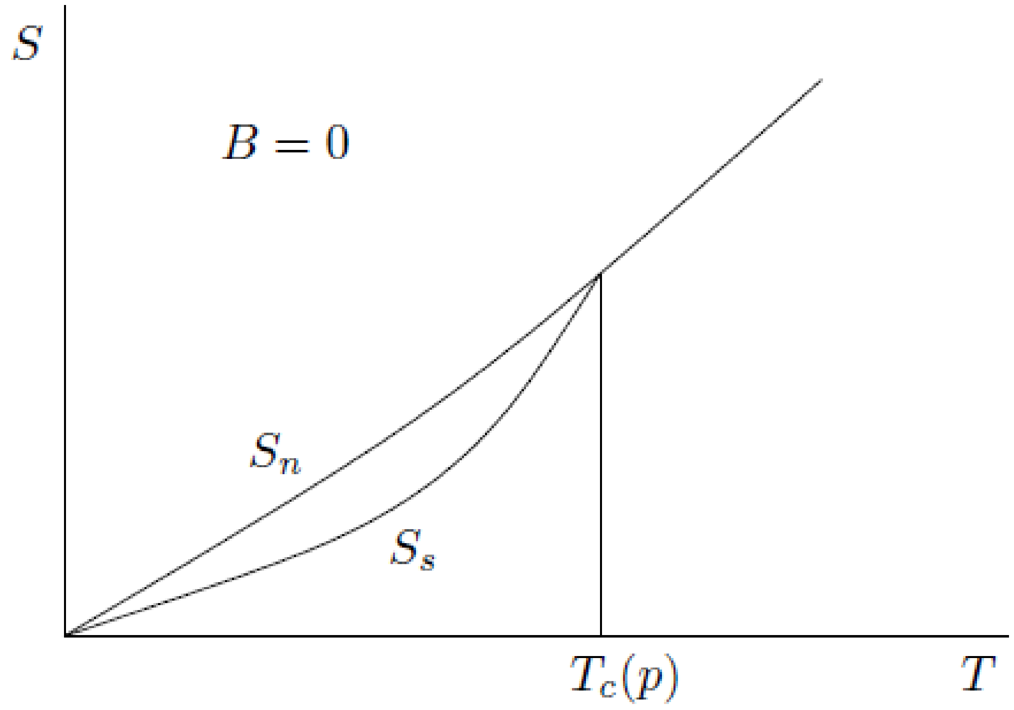
The second London equation:

$$\frac{\partial}{\partial \boldsymbol{r}} \times \left(\Lambda \boldsymbol{j}_s\right) = -\boldsymbol{B}.$$

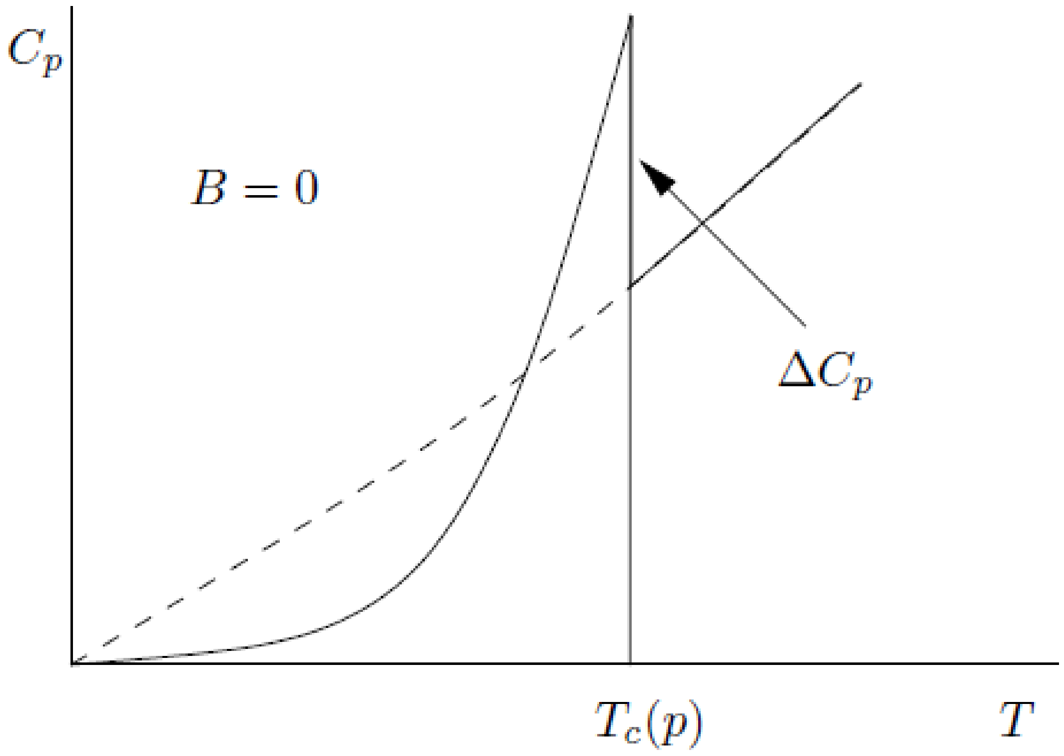
It yields the ideal diamagnetism, the Meissner effect, and the flux quantization.

Superconductor

Phase transition of superconductors:



Entropy of superconductor



Heat capacitor of superconductor

Superconductor

Now, we should introduce two more important SC theories:

Free energy form from Landau theory

1. **Landau theory**

   Ginsburg-Landau equation

$$\mathcal{F}(t,|\Psi|^2) = \mathcal{F}_n(t) + A(t)|\Psi|^2 + \frac{1}{2}B(t)|\Psi|^4 + \cdots$$

$$\frac{\partial}{\partial \boldsymbol{r}} \times \boldsymbol{B}_{\text{tot}} = \mu_0 \boldsymbol{j}_s, \qquad \boldsymbol{B}_{\text{tot}} = \boldsymbol{B} + \boldsymbol{B}_m = \frac{\partial}{\partial \boldsymbol{r}} \times \boldsymbol{A},$$

$$\boldsymbol{j}_s = \frac{ie\hbar}{2m}\left(\Psi^* \frac{\partial}{\partial \boldsymbol{r}}\Psi - \Psi \frac{\partial}{\partial \boldsymbol{r}}\Psi^*\right) - \frac{2e^2}{m}\Psi^* \boldsymbol{A}\Psi.$$

2. BCS theory

   Landau Fermi-liquid and Cooper for

Superconductor

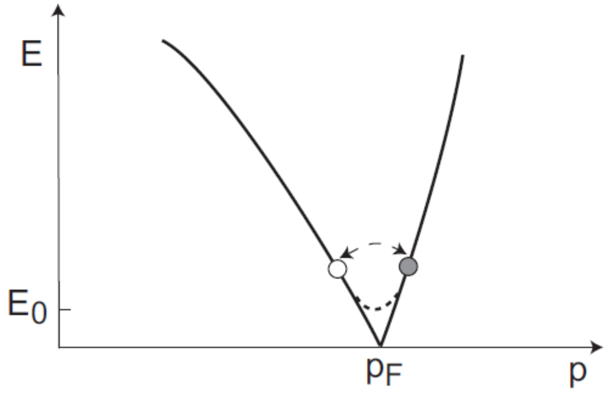
Now, we should introduce two more important SC theories:

1. Landau theory

   Ginsburg-Landau equation

2. **BCS theory**

   Landau Fermi-liquid and Cooper formation



$$\Psi^\dagger(\mathbf{r}\uparrow) = \sum_n \left[\gamma_{n\uparrow}^\dagger u_n^*(\mathbf{r}) - \gamma_{n\downarrow} v_n(\mathbf{r})\right]$$
$$\Psi^\dagger(\mathbf{r}\downarrow) = \sum_n \left[\gamma_{n\downarrow}^\dagger u_n^*(\mathbf{r}) + \gamma_{n\uparrow} v_n(\mathbf{r})\right]$$

$$\Psi(\mathbf{r}\uparrow) = \sum_n \left[\gamma_{n\uparrow} u_n(\mathbf{r}) - \gamma_{n\downarrow}^\dagger v_n^*(\mathbf{r})\right]$$
$$\Psi(\mathbf{r}\downarrow) = \sum_n \left[\gamma_{n\downarrow} u_n(\mathbf{r}) + \gamma_{n\uparrow}^\dagger v_n^*(\mathbf{r})\right]$$

$$\mathcal{H} = \mathcal{H}_{kin} + \mathcal{H}_{int}$$

$$\mathcal{H}_{kin} = \sum_\alpha \int d^3r \Psi^\dagger(\mathbf{r},\alpha)\hat{H}_e\Psi(\mathbf{r},\alpha)$$

$$\mathcal{H}_{int} = \frac{1}{2}\sum_{\alpha,\beta}\int d^3r_1\, d^3r_2 \Psi^\dagger(\mathbf{r}_2,\alpha)\Psi^\dagger(\mathbf{r}_1,\beta)W(\mathbf{r}_1,\mathbf{r}_2)\Psi(\mathbf{r}_1,\beta)\Psi(\mathbf{r}_2,\alpha)$$
$$= -\frac{W}{2}\sum_{\alpha,\beta}\int d^3r \Psi^\dagger(\mathbf{r},\alpha)\Psi^\dagger(\mathbf{r},\beta)\Psi(\mathbf{r},\beta)\Psi(\mathbf{r},\alpha)$$

$$\langle\mathcal{H}_{eff}\rangle = \mathcal{Z}^{-1}\sum_k \left\langle \psi_k^\dagger \left|\mathcal{H}_{eff}\right| \psi_k \right\rangle \exp(-E_k/T)$$

$$\mathcal{H}_{eff}\psi_k = E_k\psi_k$$

$$\mathcal{Z} = \sum_k \exp(-E_k/T)$$

Superconductor

Now, we should introduce two more important SC theories:

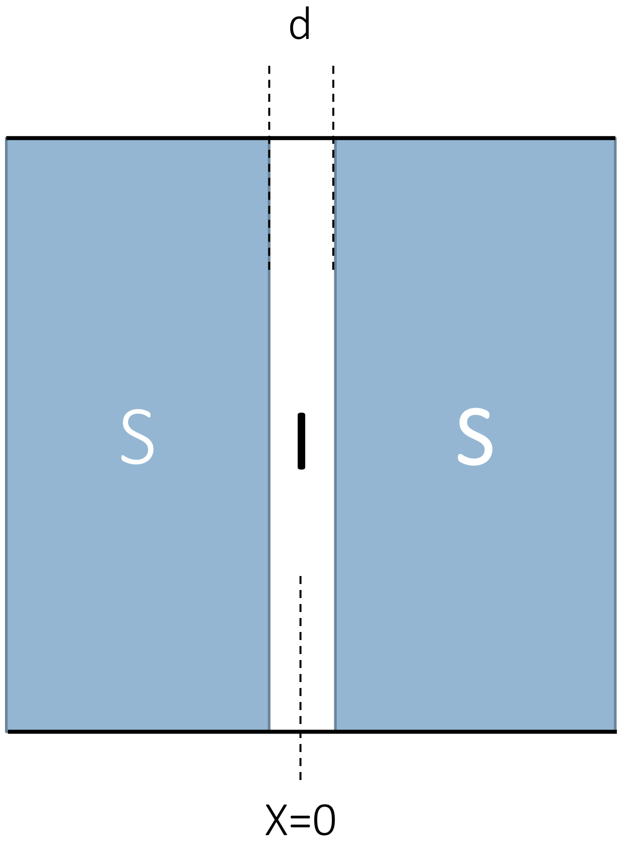
1. Landau theory
   Ginsburg-Landau equation

2. BCS theory
   Landau Fermi-liquid and Cooper formation

To understand the critical temperature and critical field of the superconductors and link to Josephson junction. And we can understand the formation of the Cooper pair leading to condensation of SC ground state.

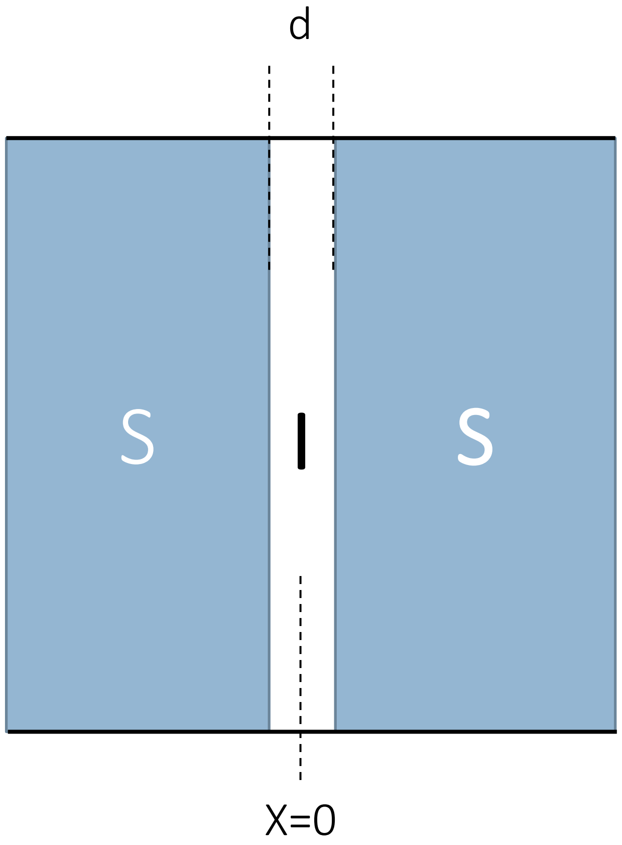
However, we need to skip them now. Please refer to Theory of Superconductivity, etc..

Josephson junction



A Josephson junction is defined as two bulk superconductors are separated by an insulator layer. The current can tunnel from one side of superconductors to another.

Josephson junction



A Josephson junction is defined as two bulk superconductors are separated by an insulator layer. The current can tunnel from one side of superconductors to another.

From Ginzburg-Landau (zero field)

$$-\frac{\hbar^2}{m^*}\nabla^2\psi + \alpha\psi + \beta|\psi|^2\psi = 0.$$

Order parameter for SC

$$|\psi_0|^2 = -\alpha/\beta$$

Ginzburg-Landau coherent length

$$\xi = \hbar/\sqrt{-2m^*\alpha}$$

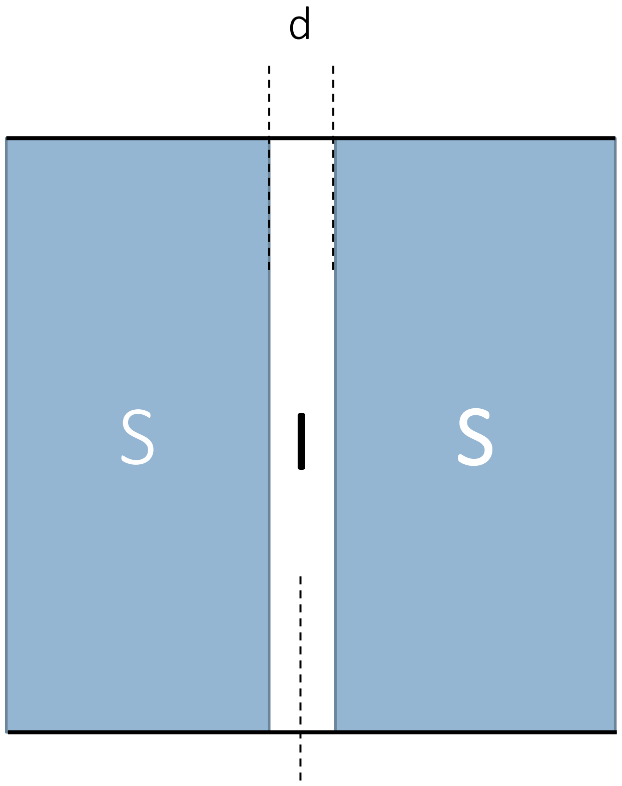
$$\xi^2\nabla^2\psi = -\psi(1 - \frac{|\psi|^2}{|\psi_0|^2}).$$

By considering $\xi >> d$, the barrier thickness and the continuity of the order parameter of three regions

$$\psi(x) = \psi_0[(\frac{1}{2} - x/d)e^{i\chi_1} + (\frac{1}{2} + x/d)e^{i\chi_2}]$$

For -d/2<x<d/2

Josephson junction



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From Ginzburg-La… …eter for SC

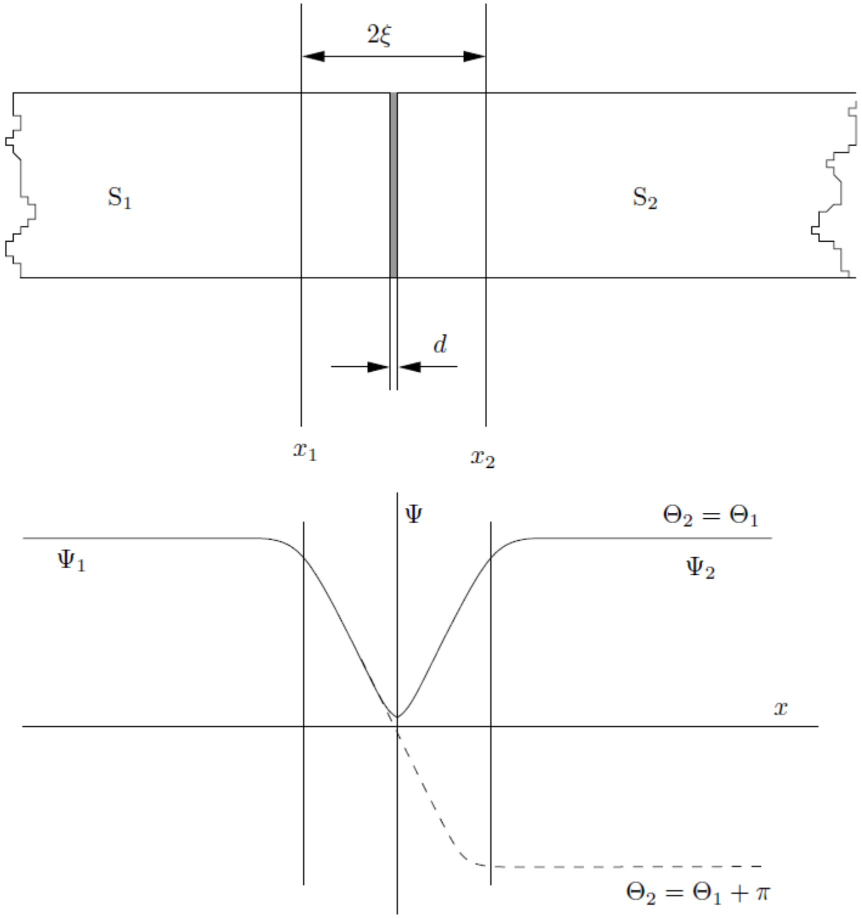
$$-\frac{\hbar^2}{m^*}\nabla^2\psi + \ldots \quad \alpha/\beta$$

Ginzburg-Landau co…

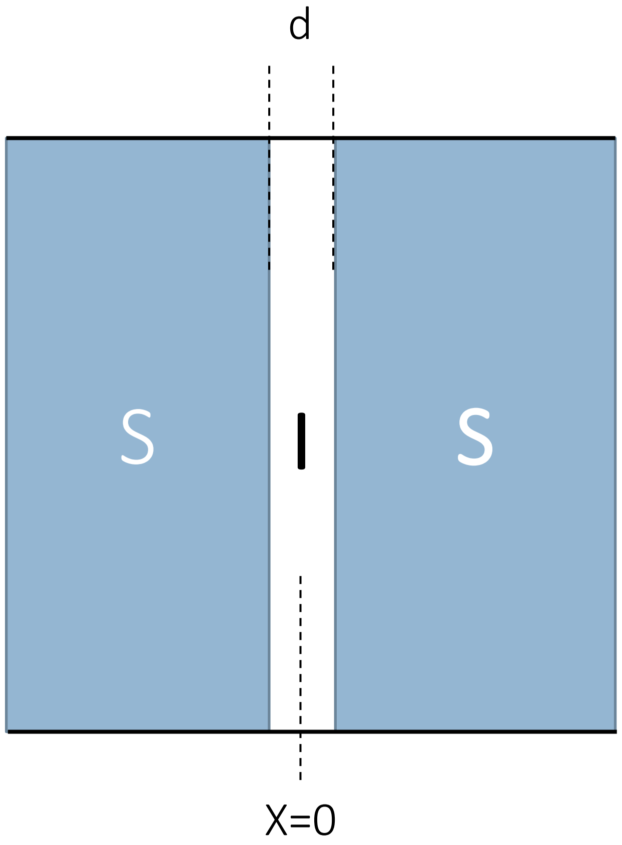
$$\xi = \hbar/\sqrt{-\ldots} \quad \psi(1 - \frac{|\psi|^2}{|\psi_0|^2}).$$

By considering $\xi >$ … …e order parameter of three regions

$\psi($… For -d/2<x<d/2



Josephson junction



Now recall another form of the Ginzburg-Landau formula

$$j = \frac{c}{4\pi}\nabla \times B = i\frac{q\hbar}{2m^*}(\psi\nabla\psi^* - \psi^*\nabla\psi) - \frac{q^2}{m^*c}|\psi|^2 A$$

Under zero magnetic field, we plugin the previous order parameter form

$$\begin{aligned} j &= i\frac{q\hbar}{2m^*}[\psi\nabla\psi^* - \psi^*\nabla\psi] \\ &= \frac{2e\hbar}{m^*d}|\psi_0|^2 \sin(\chi_2 - \chi_1). \end{aligned}$$

χ_1 and χ_2 are the phase terms of the bulk SC

This can refer to the DC Josephson relationship

Feynman's approach

Ref. Peder Heiselberg
Niels Bohr Institute

The electron's wave function in SC can be written as

$$\Psi = \sum_\alpha C_\alpha(t)\psi_\alpha \qquad \alpha = 1,2$$

This will obey the time-dependent Schrodinger's equation:

$$i\hbar\frac{d}{dt}\Psi = H\Psi$$

One can obtain the following equations:

$$i\hbar\sum_\alpha \dot{C}_\alpha\psi_\alpha + C_\alpha\dot{\psi}_\alpha = \sum_\alpha C_\alpha H\psi_\alpha$$

Multiply by ψ_β^* and integral over the space:

$$\int i\hbar\sum_\alpha \psi_\beta^*\dot{C}_\alpha\psi_\alpha + \psi_\beta^* C_\alpha\dot{\psi}_\alpha dV = \int\sum_\alpha C_\alpha\psi_\beta^* H\psi_\alpha dV$$

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Multiply by ψ_β^* and integral over the space:

$$\int i\hbar\sum_{\alpha}\psi_\beta^*\dot{C}_\alpha\psi_\alpha + \psi_\beta^* C_\alpha\dot{\psi}_\alpha dV = \int\sum_{\alpha} C_\alpha\psi_\beta^* H\psi_\alpha dV$$

Using the uncoupled condition,
The formula can be written as

$$\sum_{\alpha} i\hbar\dot{C}_\alpha\delta_{\alpha\beta} + \int\psi_\beta^* C_\alpha E_\alpha\psi_\alpha dV = \sum_{\alpha} C_\alpha H_{\beta\alpha}$$

$$\sum_{\alpha} i\hbar\dot{C}_\alpha\delta_{\alpha\beta} = \sum_{\alpha}[H_{\beta\alpha} - E_\alpha\delta_{\alpha\beta}]C_\alpha$$

Finally, we reach the following equation

$$i\hbar\frac{d}{dt}C_\beta = \sum_{\alpha}[H_{\beta\alpha} - E_\alpha\delta_{\alpha\beta}]C_\alpha$$

$$H_{\beta\alpha} = \int\psi_\beta^* H\psi_\alpha dV$$

Feynman's approach

Ref. Peder Heiselberg
Niels Bohr Institute

$$H_{\beta\alpha} = \int \psi_\beta^* H \psi_\alpha dV$$

Now, we can figure out the matrix elements of $H_{\beta\ \alpha}$

With a voltage, V across the junction, the diagonal terms

$$H_{11} = E_1 + e^*V/2 = E_1 + eV \qquad H_{22} = E_2 - e^*V/2 = E_2 - eV$$

The off-diagonal term k, represents the transition between states

$$H_{12} = H_{21} = -K$$

$$i\hbar \frac{d}{dt} C_\beta = \sum_\alpha [H_{\beta\alpha} - E_\alpha \delta_{\alpha\beta}] C_\alpha$$

Therefore, one can arrive at the equations by plugging in all the terms

$$i\hbar \frac{d}{dt} C_1 = eVC_1(t) - KC_2(t)$$

$$i\hbar \frac{d}{dt} C_2 = -eVC_2(t) - KC_1(t)$$

$$H_{\beta\alpha} = \int \psi_\beta^* H \psi_\alpha dV$$

Feynman's approach

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$$i\hbar\frac{d}{dt}C_1 = eVC_1(t) - KC_2(t)$$
$$i\hbar\frac{d}{dt}C_2 = -eVC_2(t) - KC_1(t)$$

With the coefficient to be normalized

$$|C_1|^2 = N_1 \quad C_1 = \sqrt{N_1}e^{i\chi_1}$$
$$|C_2|^2 = N_2 \quad C_2 = \sqrt{N_2}e^{i\chi_2}$$

Here N1 and N2 represent the number of superconducting electrons in the SC1 and SC2. We can arrive at the following equation:

$$i\hbar\frac{d}{dt}\sqrt{N_1}e^{i\chi_1} = eV\sqrt{N_1}e^{i\chi_1} - K\sqrt{N_2}e^{i\chi_2}$$

Multiply by C1

$$i\hbar[\frac{d}{dt}N_1 + i2N_1\frac{d}{dt}\chi_1] = 2eVN_1 - 2K\sqrt{N_1N_2}e^{i(\chi_2-\chi_1)}$$

Feynman's approach

Ref. Peder Heiselberg
Niels Bohr Institute

So, we can do the same for two equations:

$$i\hbar[\frac{d}{dt}N_1 + i2N_1\frac{d}{dt}\chi_1] = 2eVN_1 - 2K\sqrt{N_1N_2}e^{i(\chi_2-\chi_1)}$$

$$i\hbar[\frac{d}{dt}N_2 + i2N_2\frac{d}{dt}\chi_2] = -2eVN_2 - 2K\sqrt{N_1N_2}e^{i(\chi_1-\chi_2)}$$

Now, we compare the real and imaginary parts:

Real part

$$\hbar N_1\frac{d}{dt}\chi_1 = -eVN_1 + K\sqrt{N_1N_2}\cos(\phi)$$

$$\hbar N_2\frac{d}{dt}\chi_2 = eVN_2 + K\sqrt{N_1N_2}\cos(\phi)$$

Imaginary part

$$\hbar\frac{d}{dt}N_1 = -2K\sqrt{N_1N_2}\sin(\phi)$$

$$\hbar\frac{d}{dt}N_2 = +2K\sqrt{N_1N_2}\sin(\phi)$$

$$\Rightarrow \quad \hbar\frac{d}{dt}N_1 = -\hbar\frac{d}{dt}N_2$$

N_1+N_2=const. Conversation of the total charges

Feynman's approach

Ref. Peder Heiselberg
Niels Bohr Institute

Still from the imaginary part

$$\hbar \frac{d}{dt} N_1 = -2K\sqrt{N_1 N_2}\sin(\phi)$$
$$\hbar \frac{d}{dt} N_2 = +2K\sqrt{N_1 N_2}\sin(\phi)$$

N_1=const.-N_2

You subtract the bottom equation with the top equation

$$\frac{d}{dt}[N_2 - N_1] = \frac{4K\sqrt{N_1 N_2}}{\hbar}\sin(\phi) \quad \xrightarrow{\text{xe}} \quad 2e\frac{d}{dt}[N_2] = \frac{4eK\sqrt{N_1 N_2}}{\hbar}\sin(\phi)$$

So, we can tell that the left side of the equation is the number of Cooper pairs flowing as the time which resembles the supercurrent density similar to the Ginzburg- Landau theory, Therefor we can define the supercurrent equation as the following:

$$I_s = I_c \sin(\phi) \qquad I_c = \frac{4eK\sqrt{N_1 N_2}}{\hbar}$$

Known as DC Josephson relationship

Feynman’s approach

Ref. Peder Heiselberg
Niels Bohr Institute

$$\hbar N_1 \frac{d}{dt}\chi_1 = -eVN_1 + K\sqrt{N_1N_2}\cos(\phi)$$
$$\hbar N_2 \frac{d}{dt}\chi_2 = eVN_2 + K\sqrt{N_1N_2}\cos(\phi)$$

Divided the top equation with N1 and the bottom equation with N2 then subtract one from another

$$\hbar\frac{d}{dt}\phi = 2eV - K\frac{N_2 - N_1}{\sqrt{N_1N_2}}\cos(\phi)$$

In general, the last term can be ignored due to the small K and phase oscillation cosine term.
We then can arrive at a simpler equations

$$\hbar\frac{d}{dt}\phi = 2eV$$

This equation is known as the AC Josephson relationship

Feynman's approach

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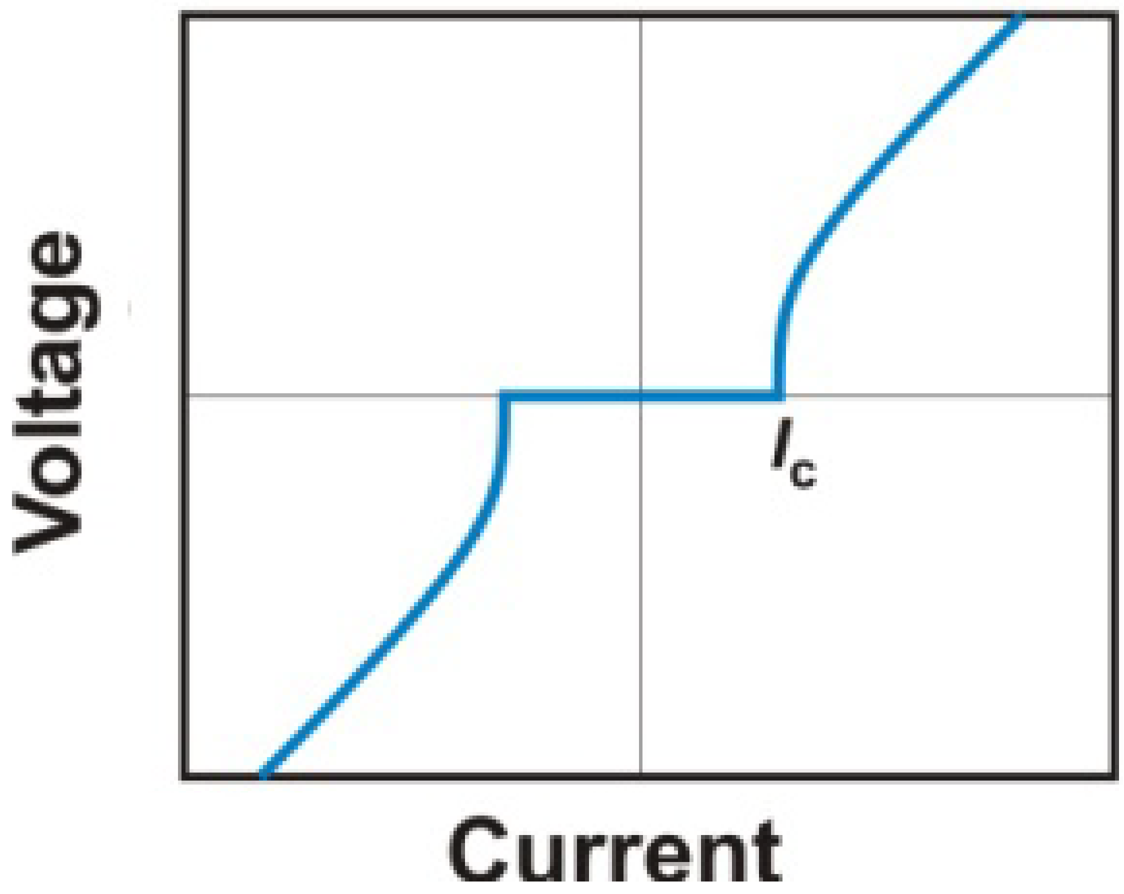
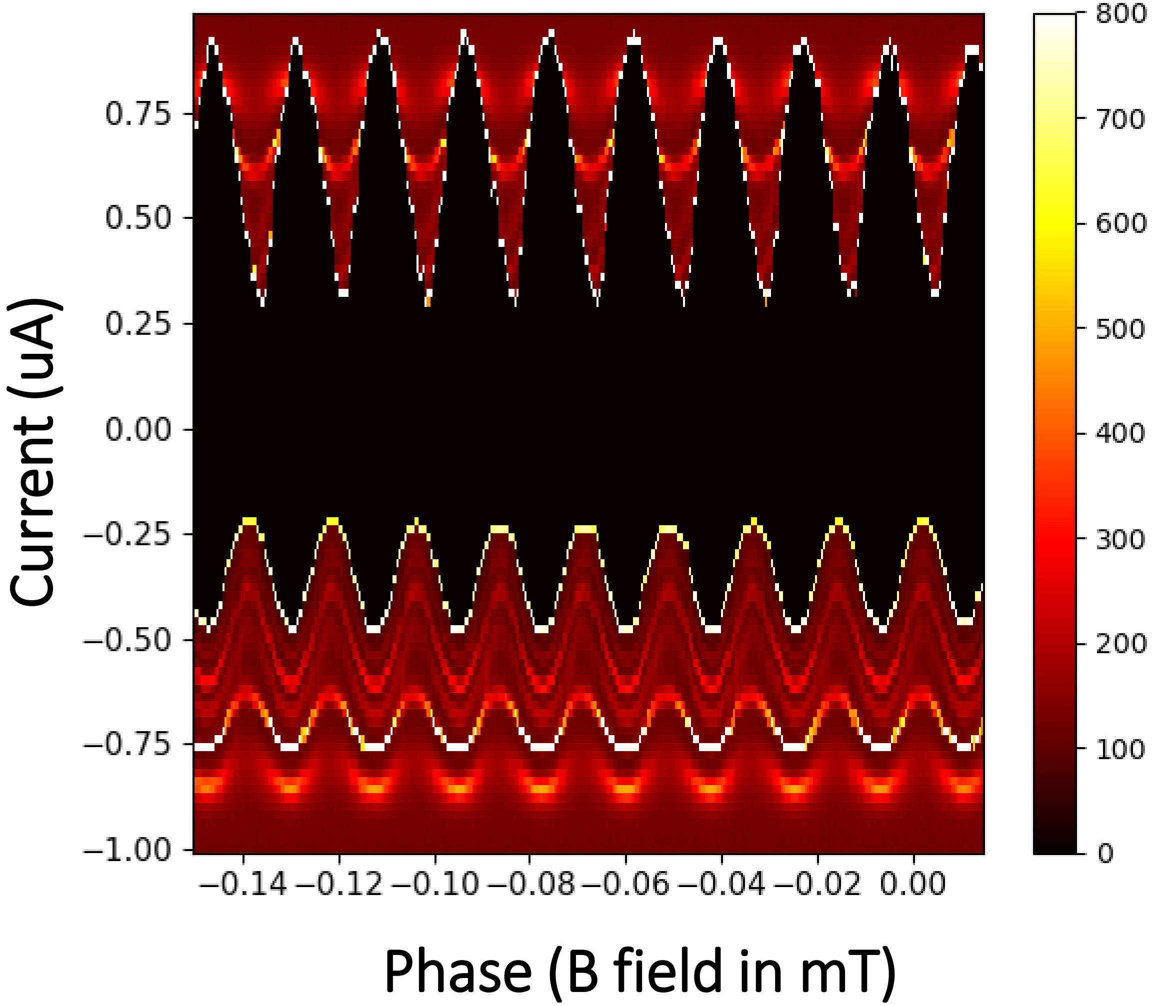
Niels Bohr Institute

DC Josephson effect, IV:

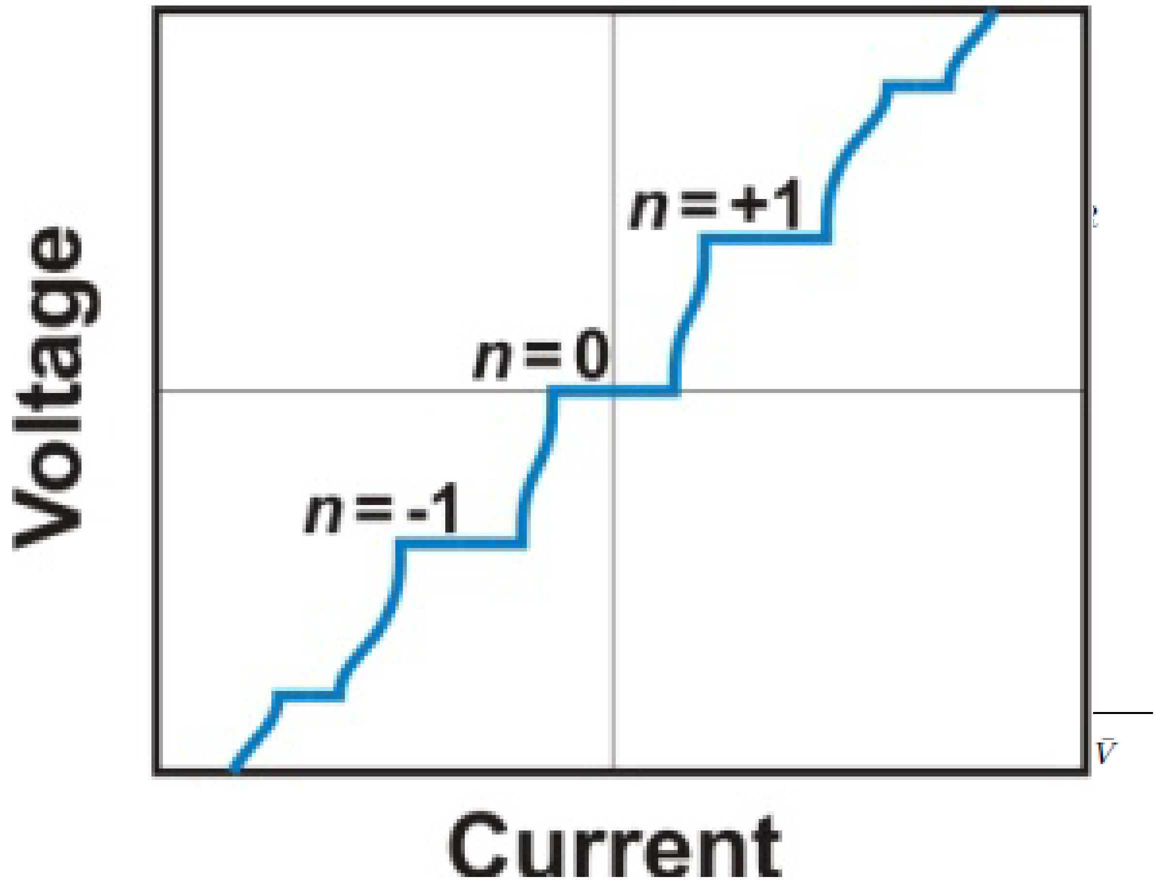


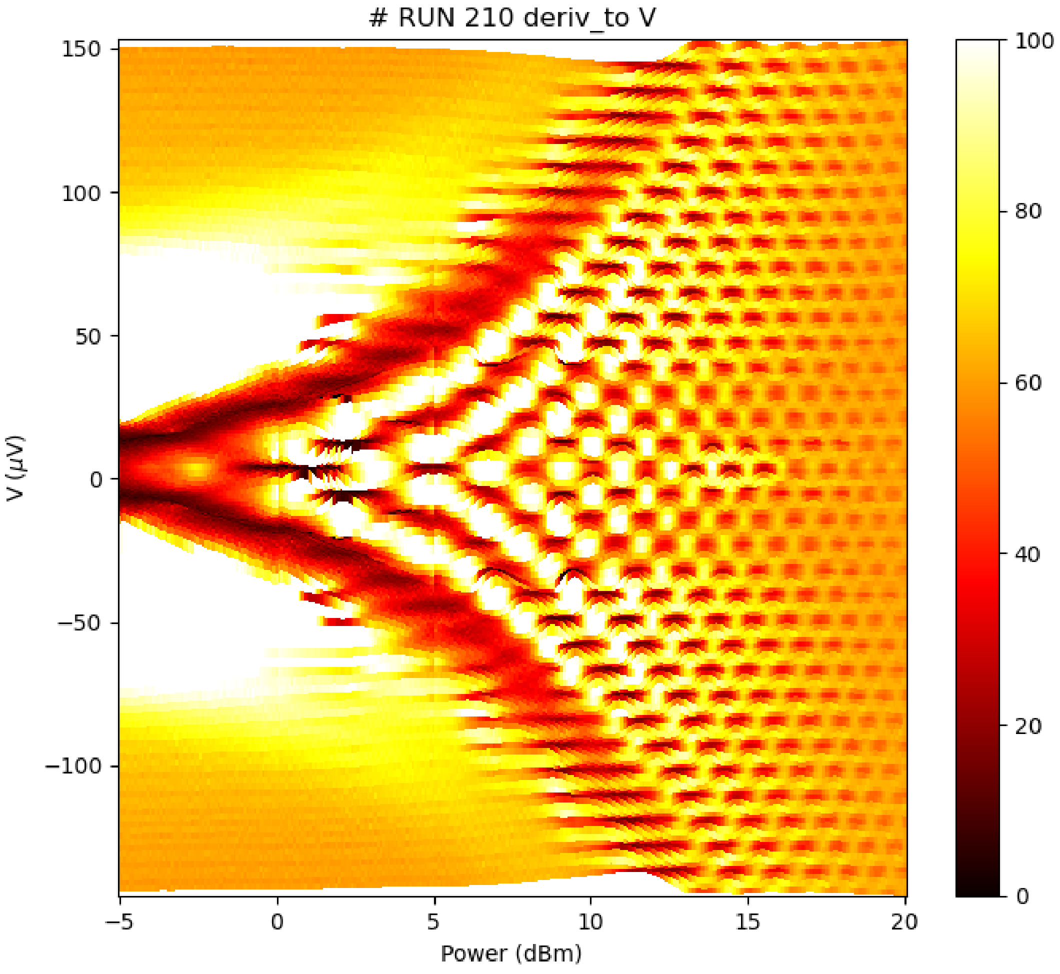


Feynman's approach

Ref. Peder Heiselberg
Niels Bohr Institute

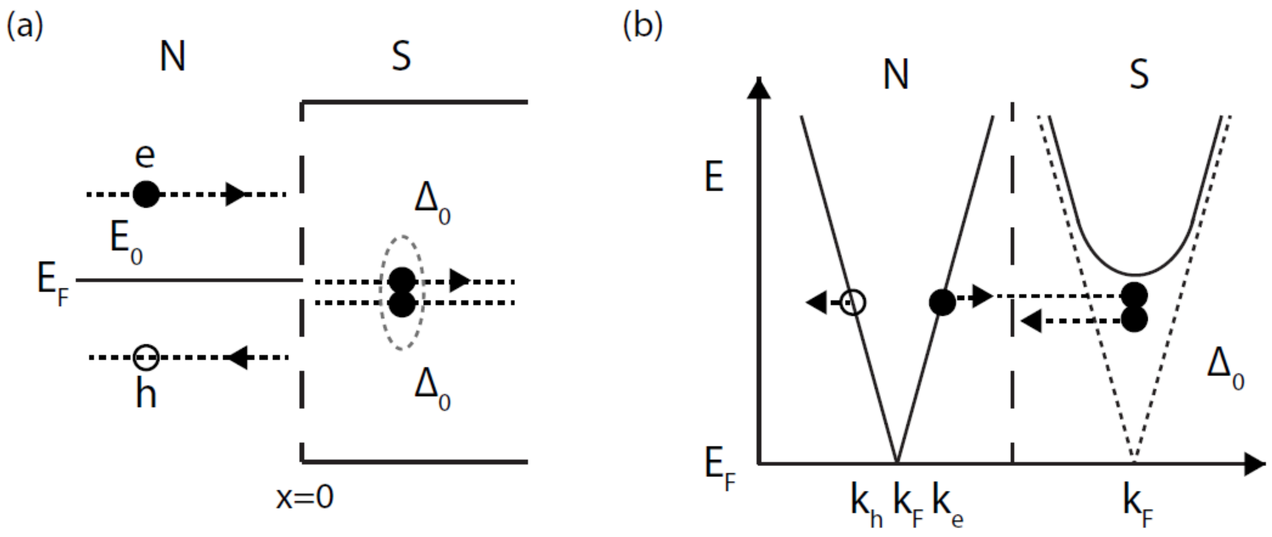
AC Josephson effect, Shapiro steps:





A SN interface

Ref. : N. B. Kopnin. Theory of superconductivity



In the previous discussion, the two superconductors are connected by a weak link or a thin insulating layer. Therefore, cooper pair tunneling through the insulator barrier. Now if we replace the insulating barrier with a normal metal. We will have an interesting behavior at the SN interface.

Electrons propagate toward the SN interface and try to enter the superconductor. To make electrons into the superconducting condensate, forming a cooper pair, an electron has to pair with another electron coming from the quasiparticle states. Therefore, the corresponding hole will have to leave the superconductor to enter the normal region. This process is called **Andreev reflection**.

The BdG formalism

We first consider the Bogoliubov de Gennes (BdG) equations based on the BCS model:

$$-\frac{\hbar^2}{2m}(\nabla - \frac{ie}{\hbar c}A)^2 u - E_F u + \Delta v = \epsilon u$$

$$\frac{\hbar^2}{2m}(\nabla + \frac{ie}{\hbar c}A)^2 v - E_F v + \Delta^* u = \epsilon v$$

where A is the vector potential, E_F is the Fermi energy, Δ is the complex coupling strength for quasiparticles, and ε is the energy for the quasiparticles. u and υ are quasiparticle wave functions for positive (e) and negative (h) energy

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Assuming wave function is slowly varying position.

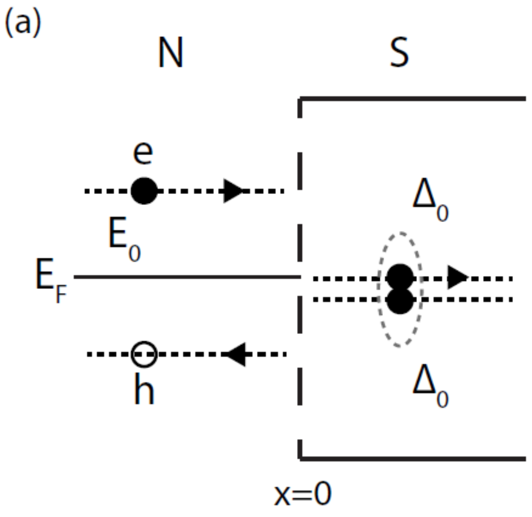
$$\begin{pmatrix} u \\ v \end{pmatrix} = e^{(i\mathbf{k}\cdot\mathbf{r})} \begin{pmatrix} U(x) \\ V(x) \end{pmatrix}$$

The BdG equations with zero magnetic field:

$$-\hbar V_F \frac{dU}{dx} + \Delta V = \epsilon U$$

$$-\hbar V_F \frac{dV}{dx} + \Delta^* U = \epsilon V$$

The BdG formalism

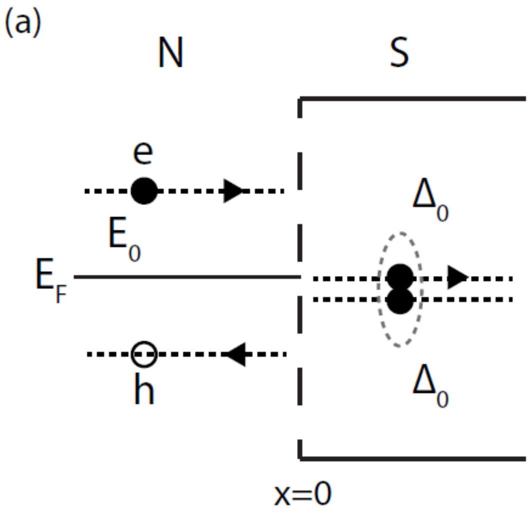


For x<0

$$\begin{pmatrix} U(x) \\ V(x) \end{pmatrix} = \begin{pmatrix} e^{(i\lambda_N x)} \\ ae^{(-i\lambda_N x)} \end{pmatrix}$$

$$\lambda_N = \epsilon/\hbar V_F$$

The BdG formalism



For x>0 and ε>Δ

$$\begin{pmatrix} U(x) \\ V(x) \end{pmatrix} = c e^{(i\lambda_s x)} \begin{pmatrix} U_0 \\ V_0 \end{pmatrix}$$

$$\lambda_S = \sqrt{\epsilon^2 . |\Delta^2|} / \hbar V_F$$

$$U_0 = \frac{1}{\sqrt{2}} (1 + \frac{\sqrt{\epsilon^2 . |\Delta|^2}}{\epsilon})^{1/2}$$

$$\text{and} \quad V_0 = \frac{1}{\sqrt{2}} (1 - \frac{\sqrt{\epsilon^2 . |\Delta|^2}}{\epsilon})^{1/2}$$

$$a = U_0 / V_0, \quad c = 1/U_0$$

For x<0

$$\begin{pmatrix} U(x) \\ V(x) \end{pmatrix} = \begin{pmatrix} e^{(i\lambda_N x)} \\ a e^{(-i\lambda_N x)} \end{pmatrix}$$

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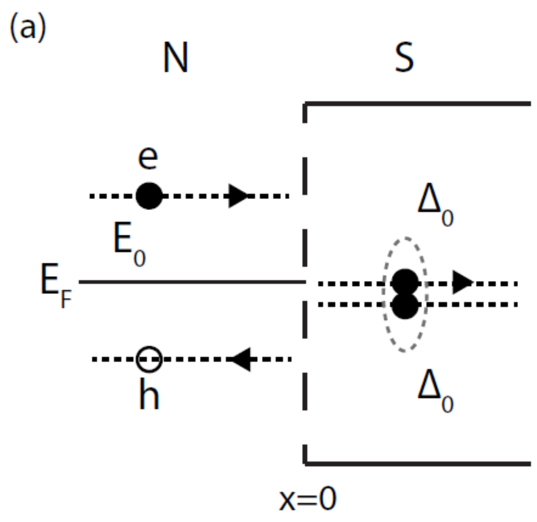
The BdG formalism



For x>0 and ε>Δ

$$\begin{pmatrix} U(x) \\ V(x) \end{pmatrix} = ce^{(i\lambda_s x)} \begin{pmatrix} U_0 \\ V_0 \end{pmatrix}$$

$$\lambda_S{=}\sqrt{\epsilon^2.|\Delta^2|}/\hbar \mathrm{V}_F$$

$$U_0 = \frac{1}{\sqrt{2}}(1 + \frac{\sqrt{\epsilon^2.|\Delta|^2}}{\epsilon})^{1/2}$$

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$$a = U_0/V_0, \quad c = 1/U_0$$

For x>0 and ε<Δ

$$\begin{pmatrix} U(x) \\ V(x) \end{pmatrix} = ce^{(-\lambda'_s x)} \begin{pmatrix} U'_0 \\ V'_0 \end{pmatrix}$$

$$\lambda'_S{=}\sqrt{|\Delta^2| - \epsilon^2}/\hbar \mathrm{V}_F$$

$$U'_0 = \frac{1}{\sqrt{2}}(1 + \frac{\sqrt{|\Delta|^2.\epsilon^2}}{\epsilon})^{1/2}$$

$$\text{and} \quad V'_0 = \frac{1}{\sqrt{2}}(1 - \frac{\sqrt{|\Delta|^2.\epsilon^2}}{\epsilon})^{1/2}$$

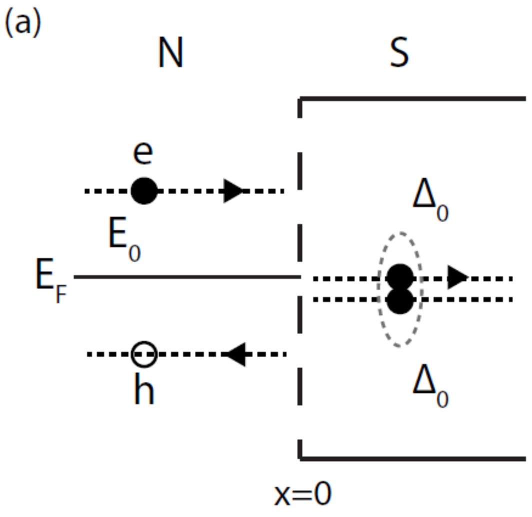
$$a = U'_0/V'_0, \quad c = 1/U'_0$$

For x<0

$$\begin{pmatrix} U(x) \\ V(x) \end{pmatrix} = \begin{pmatrix} e^{(i\lambda_N x)} \\ ae^{(-i\lambda_N x)} \end{pmatrix}$$

$$\lambda_N{=}\epsilon/\hbar \mathrm{V}_F$$

The BdG formalism



For x>0 and ε>Δ

$$a = U_0/V_0, \quad c = 1/U_0$$

For x>0 and ε<Δ

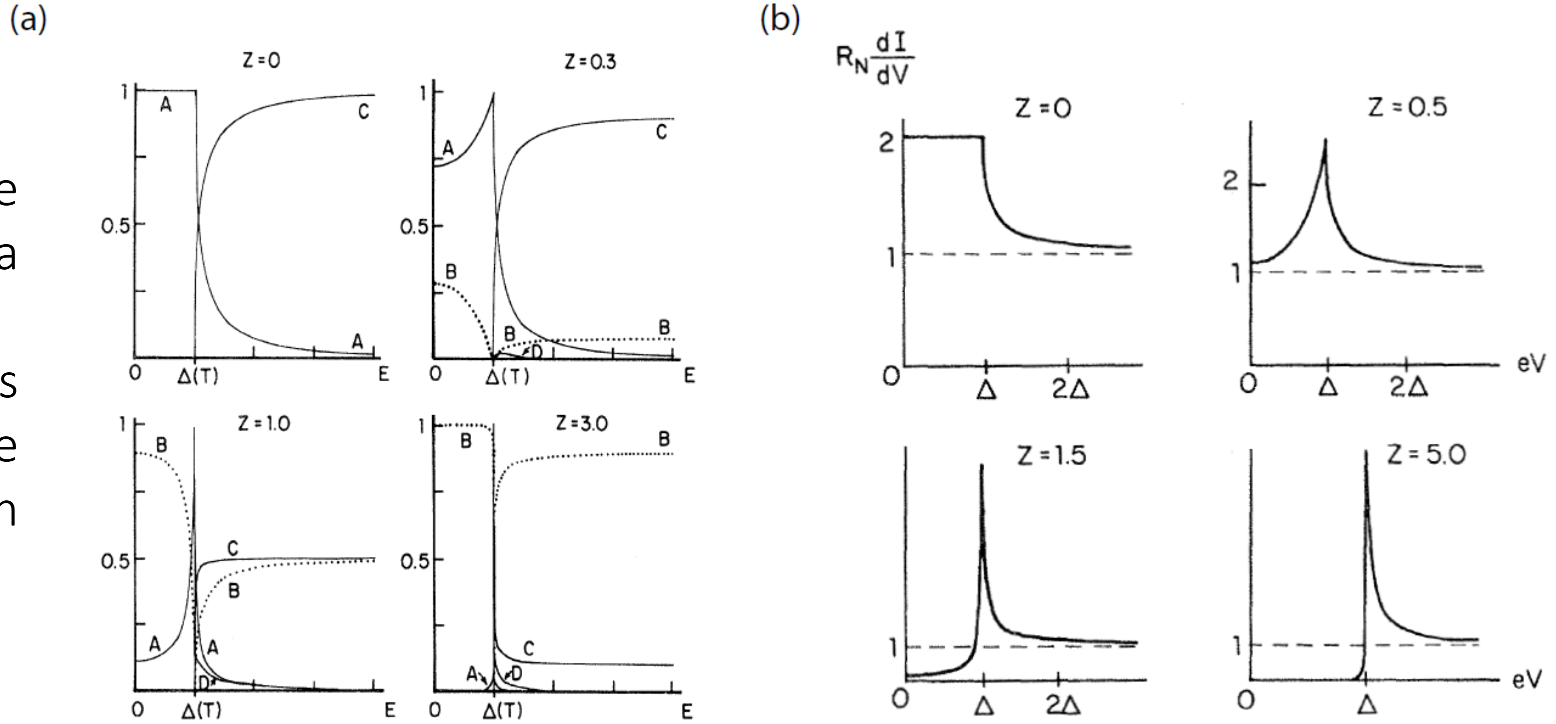
$$a = U_0'/V_0', \quad c = 1/U_0'$$

In a perfect case, $|a|^2=1$ meaning the conversion of electron and hole is 100%. Therefore, one obtains a perfect Andreev reflection.

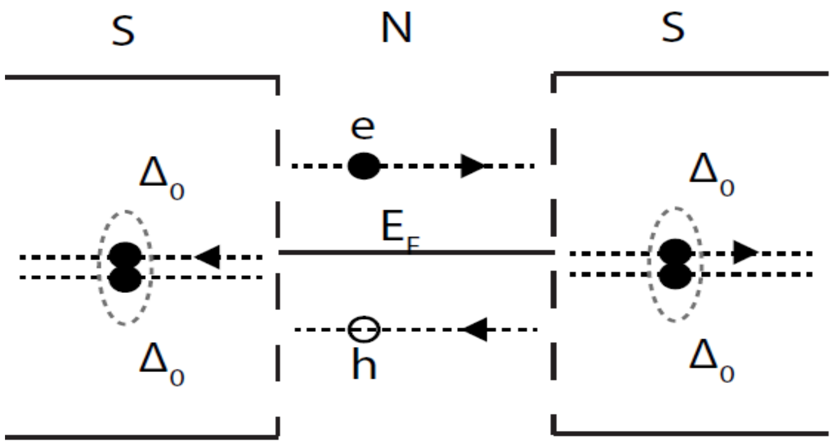
With the help of the BTK model, we can analyze the NS interface with a reflection.

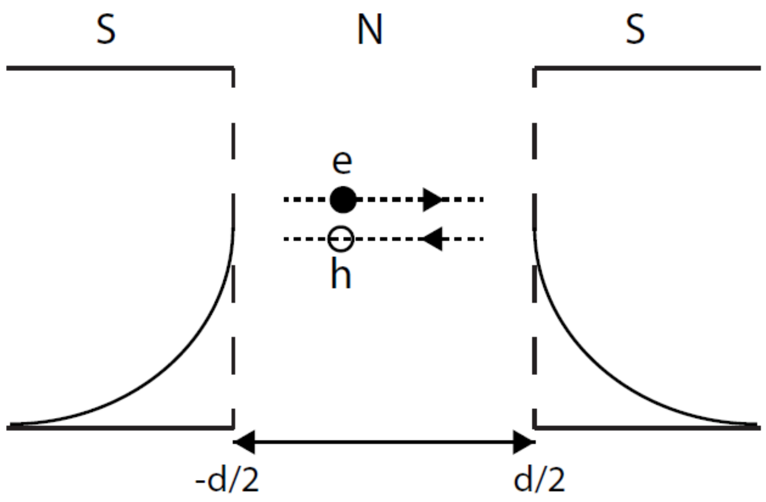
Here one can define a dimensionless barrier height Z that can be considered as a delta function between N and S interface.

Here $T=1/(1+Z^2)$ and $R=Z^2/(1+Z^2)$



A SNS Josephson junction





Now, a barrier region is formed with a normal material sandwiched between two superconductors. Between two superconductors, Andreev bond states will be established.

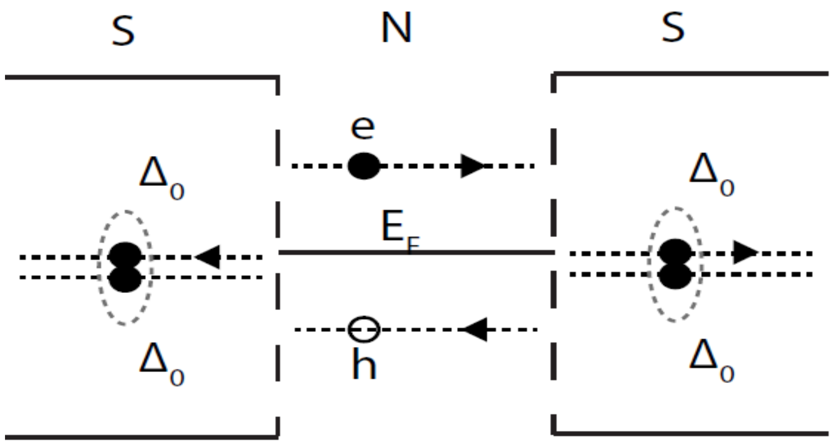
Similarly, we can apply the previous BdG formalism to analyze three regions.

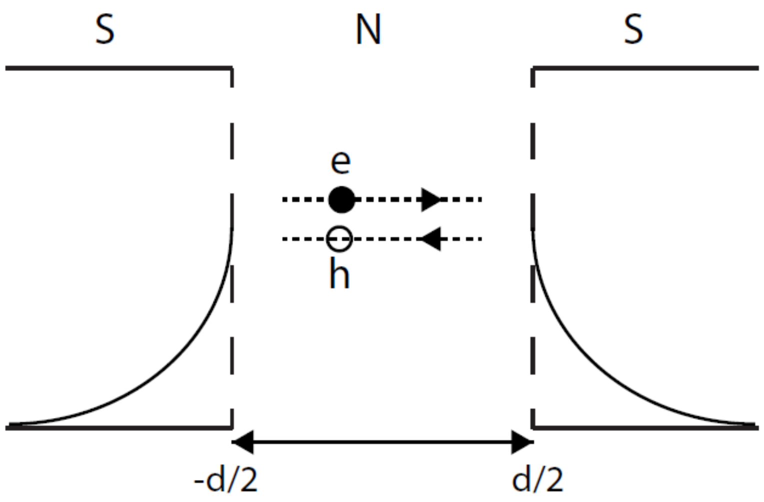
$$1)\ x > d/2: \quad \begin{pmatrix} U(x) \\ V(x) \end{pmatrix} = d_1' e^{\lambda_s' x} \begin{pmatrix} U_0' e^{-i\phi/4} \\ V_0' e^{i\phi/4} \end{pmatrix}$$

$$2)\ d/2 > x > -d/2: \quad \begin{pmatrix} U(x) \\ V(x) \end{pmatrix} = A \begin{pmatrix} e^{i\lambda_N x} \\ a e^{i\lambda_N x} \end{pmatrix}$$

$$3)\ x < -d/2: \quad \begin{pmatrix} U(x) \\ V(x) \end{pmatrix} = d_1' e^{-\lambda_s' x} \begin{pmatrix} U_0' e^{i\phi/4} \\ V_0' e^{-i\phi/4} \end{pmatrix}$$

A SNS Josephson junction





A can be found by matching the boundary conditions and normalization of the wave function.

$$|A|^2 = \frac{1}{2}\left(\frac{\sqrt{|\Delta|^2+\epsilon^2}}{\hbar|V_F| + d\sqrt{|\Delta|^2+\epsilon^2}}\right)$$

With quasiparticle energy:

$$\epsilon = \pm\frac{\hbar}{|\omega_x|}\left(\frac{\phi}{2} \mp arcsin\left(\frac{\epsilon}{|\Delta|} + \pi\left(l \pm \frac{1}{2}\right)\right)\right)$$

Where ω_x=V_F/d and l is the integer.
We can compare two limits:

1. short junction — d<< ξ_0 $(\omega_x >> |\Delta|)$
2. long junction — d>> ξ_0 $(\omega_x << |\Delta|)$

A SNS Josephson junction

Where $\omega_x = V_F/d$ and l is the integer. We can compare two limits:

1. short junction

$$d << \xi_0 \ (\omega_x >> |\Delta|)$$

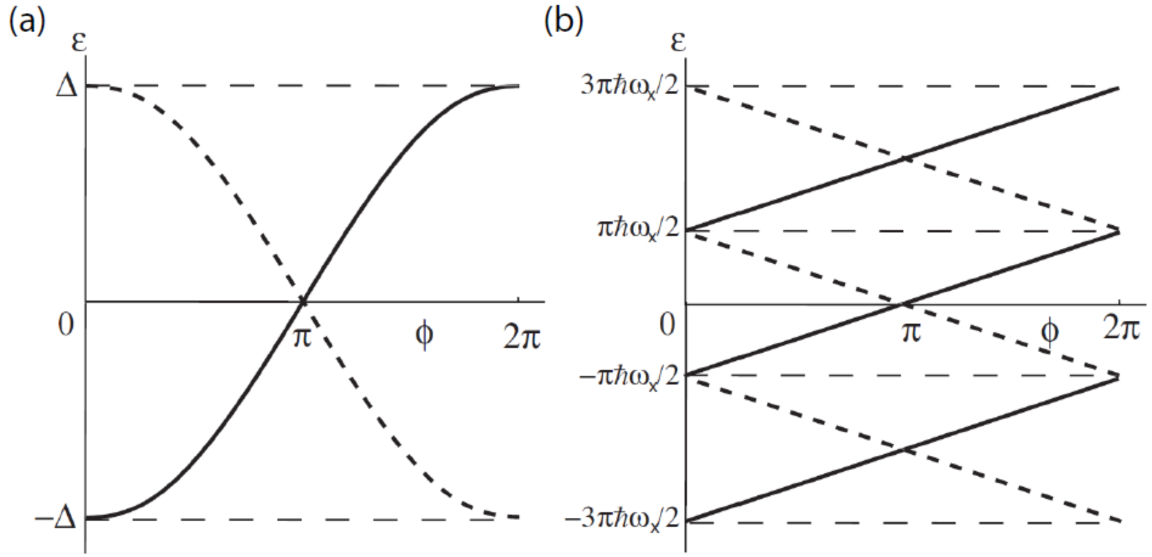
$$\epsilon = \mp|\Delta|cos(\frac{\phi}{2})$$

2. long junction

$$d >> \xi_0 \ (\omega_x << |\Delta|)$$

$$\epsilon = \pm\hbar|\omega_x|(\frac{\phi}{2} - \frac{\pi}{2}) + \pi\hbar|\omega_x|M$$

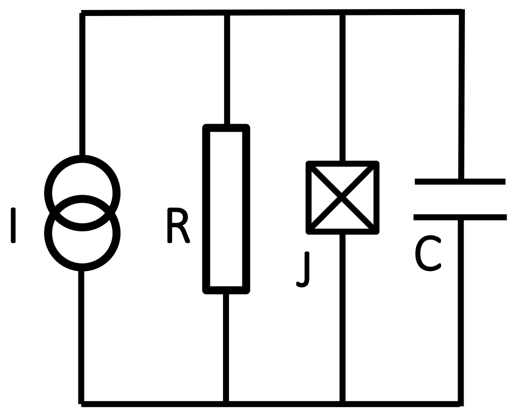
For the quasiparticle spectrum and M is an integer



For short and long, the dominated energy is by Δ and $\hbar\omega_x$, respectively. That will correspond to the superconducting gap and Thouless energy for the ballistic case.

D. E. McCumber JAP (1968), M. Tinkham, Introduction to superconductivity
Peder Heiselberg, Niels Bohr Institute

RCSJ model



Here we use the resistor-capacitor shunt junction circuit to model the current flow in the Josephson junction. There are three currents based on Kirchhoff's law:

1. Josephson current $I_c \sin(\phi)$
2. Displacement current $\frac{\hbar C}{2e}\frac{\partial^2 \phi}{\partial t^2}$
3. Dissipative(quasiparticle) current $\frac{\hbar}{2eR}\frac{\partial \phi}{\partial t}$

Relationship in displacement C

$$I_C = C\frac{\partial V}{\partial t} = \frac{\hbar C}{2e}\frac{\partial^2 \phi}{\partial t^2}$$

Therefore, we can have a total current I

$$I = \frac{\hbar C}{2e}\frac{\partial^2 \phi}{\partial t^2} + \frac{\hbar}{2eR}\frac{\partial \phi}{\partial t} + I_c \sin(\phi)$$

Or rewritten as

$$\frac{\hbar C}{2e}\frac{\partial^2 \phi}{\partial t^2} + \frac{\hbar}{2eR}\frac{\partial \phi}{\partial t} = \boxed{I - I_c \sin(\phi)}$$

The right-hand side is the force term from the pendulum from the potential.

RCSJ model

D. E. McCumber JAP (1968), M. Tinkham, Introduction to superconductivity
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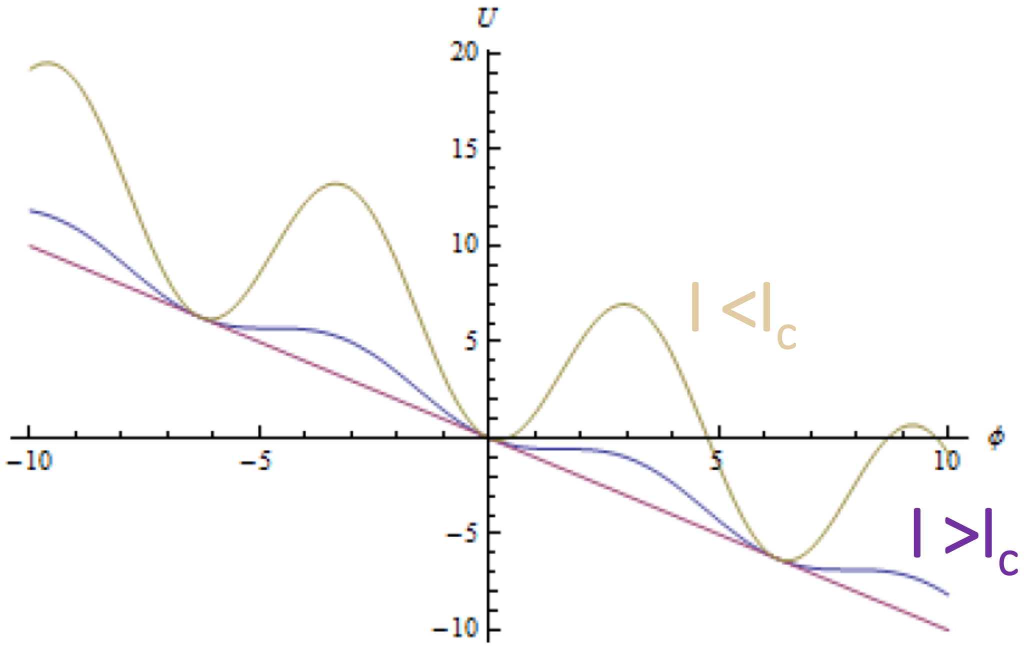
Potential can be integral from the right hand side:

$$U = -\int (I - I_c \sin(\phi)) d\phi = -I\phi - I_c \cos(\phi) + k$$

$$U = I_c[1 - \cos(\phi)] - I\phi = I_c \left[1 - \cos(\phi) - \phi \frac{I}{I_c}\right]$$

This is the tilted washboard potential.



Another way of writing the equation :

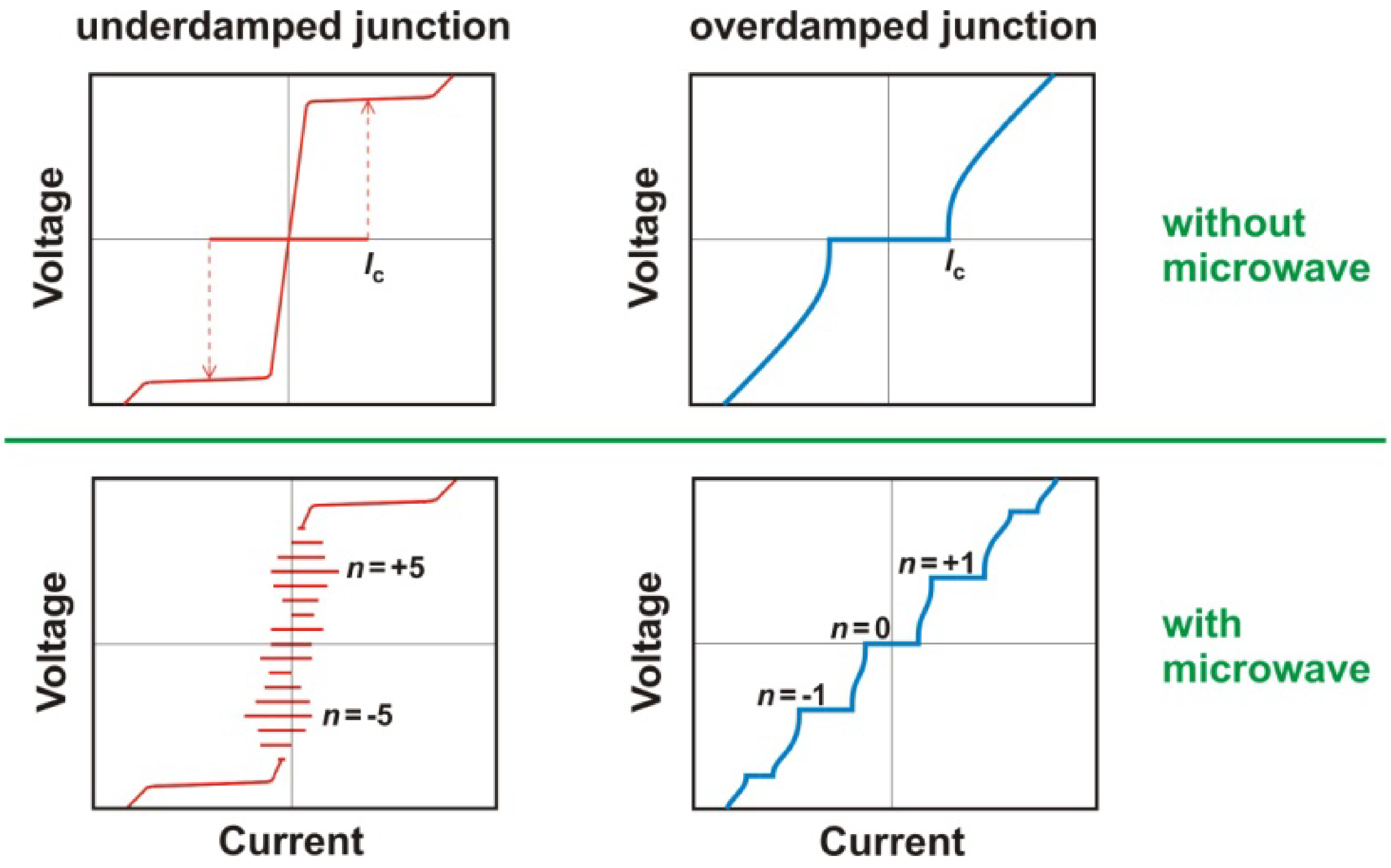
$$F = J\frac{d^2\phi}{dt^2} + \eta\frac{d\phi}{dt} + E_J sin(\phi)$$

$F = \frac{\hbar I}{2e}$ $J = \frac{\hbar^2 C}{4e^2}$, $\eta = \frac{\hbar^2}{4e^2 R}$ and $E_J = \frac{\hbar I_{C0}}{2e}$

We will reach to the same potential :

$$U(\phi) = E_J(1 - cos(\phi)) - F\phi = E_J(1 - cos(\phi) - \frac{I}{I_{C0}}\phi)$$

RCSJ model



$$\beta_C \equiv \frac{2\pi}{\Phi_0} I_0 R^2 C$$

strong damping: friction term $\dot{\delta}$ dominates, i.e. $\beta_C \ll 1$

particle gets retrapped at $I=I_0$

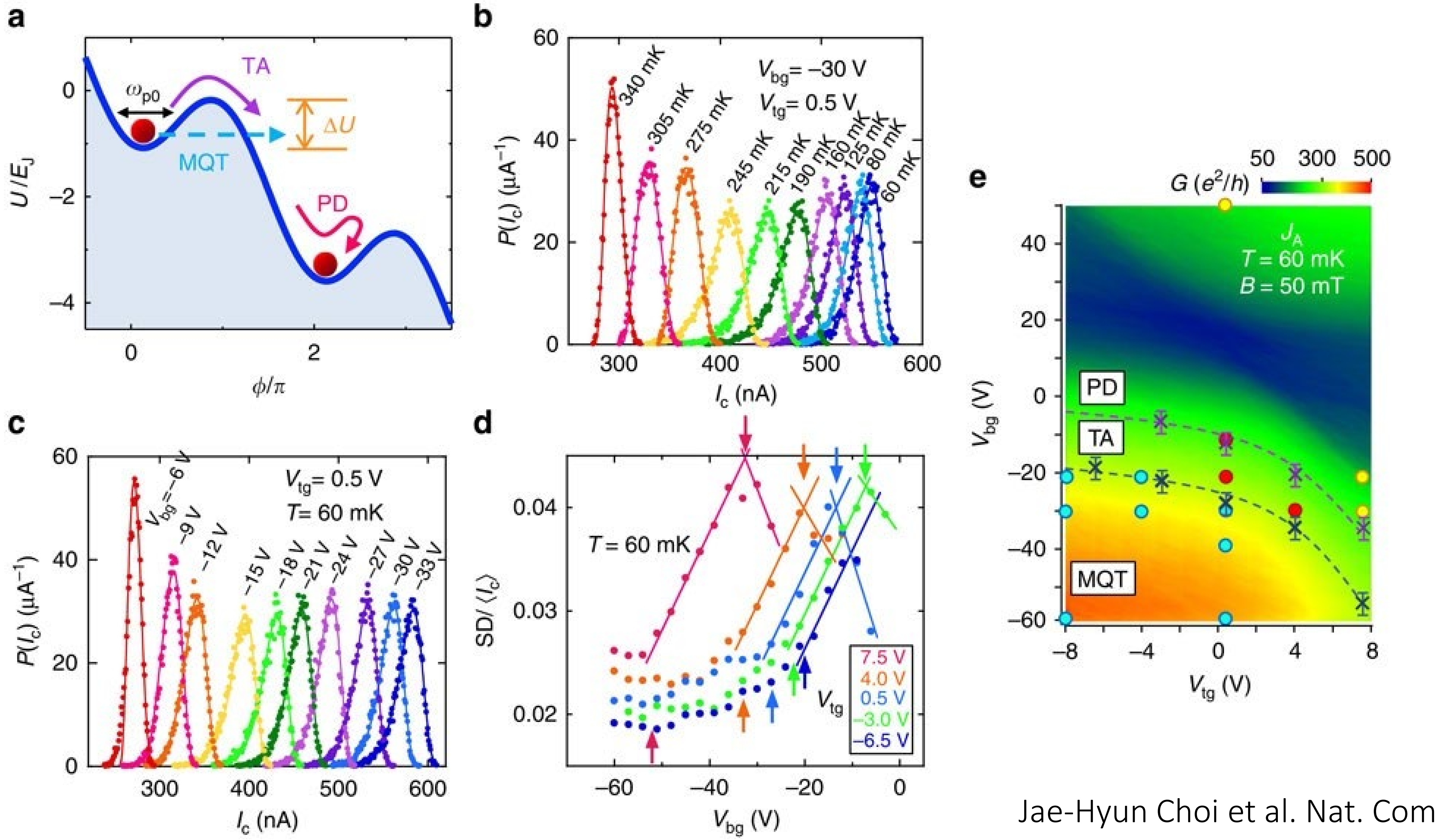
⇨ non-hysteretic IVC

weak damping: inertial term $\ddot{\delta}$ dominates, i.e. $\beta_C \gg 1$

particle gets retrapped at $I=I_r<I_0$

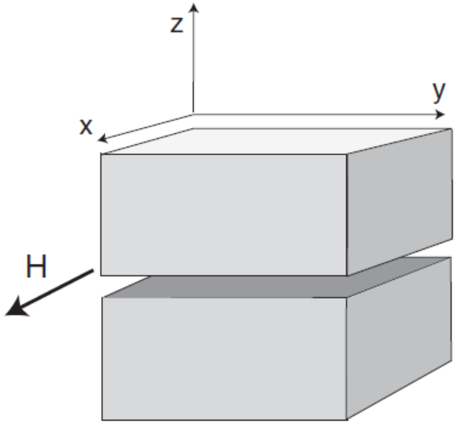
⇨ hysteretic IVC

RCSJ model



Jae-Hyun Choi et al. Nat. Commun. 2013

Josephson junction under a magnetic field



By choosing a vector potential

$$h_x = -\frac{\partial A_y}{\partial z}$$

$$h_x(z) = \begin{cases} He^{-(z-d/2)/\lambda_2} & z > d/2 \\ He^{(z+d/2)/\lambda_1} & z < -d/2 \end{cases}$$

$$\frac{\partial \phi}{\partial y} = -\frac{2e(\lambda_1 + \lambda_2 + d)H}{\hbar c}$$

We can reach the relationship between phase and magnetic field.

Using Maxwell's equation, we can reach the following results

$$\lambda_J^2 \frac{\partial^2 \phi}{\partial y^2} = \sin\phi \qquad \lambda_J = \sqrt{\frac{\hbar c^2}{8\pi e j_c(\lambda_2 + \lambda_1 + d)}}$$

By integral with $\partial\phi/\partial y$

$$\frac{\lambda_J^2}{2}\left(\frac{\partial \phi}{\partial y}\right)^2 + \cos\phi = A$$

Let me skip a few steps

$$j_s = j_c \sin\left(\frac{2\pi y}{L} + \phi_0\right)$$

$$L = \frac{8\pi^2 j_c \lambda_J^2}{cH} = \frac{\Phi_0}{(\lambda_1 + \lambda_2 + d)H}$$

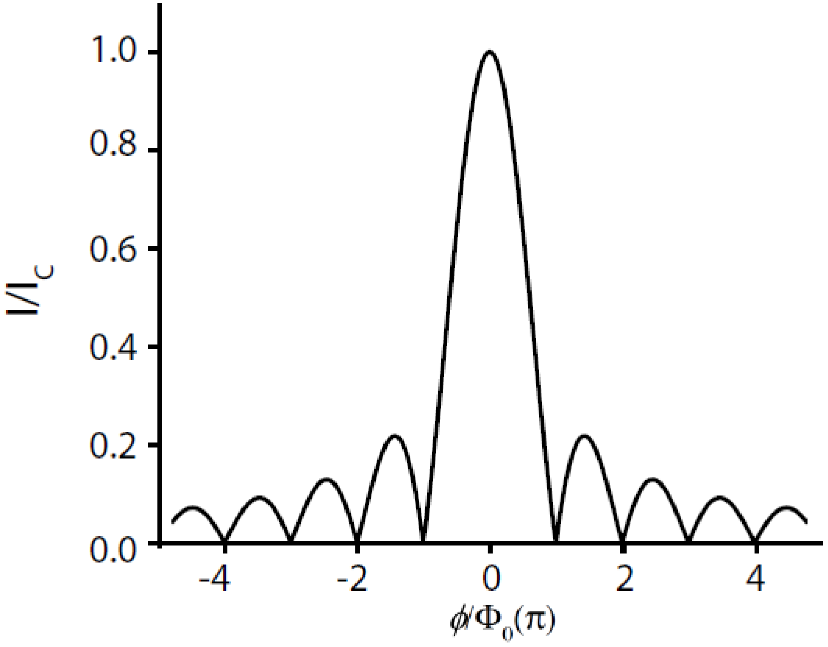
$$I = j_c \int_{-W/2}^{W/2} \sin\left(\frac{2\pi y}{L} + \phi_0\right) dy = W j_c \frac{\sin(\pi W/L)}{\pi W/L} \sin\phi_0$$

Josephson junction under a magnetic field

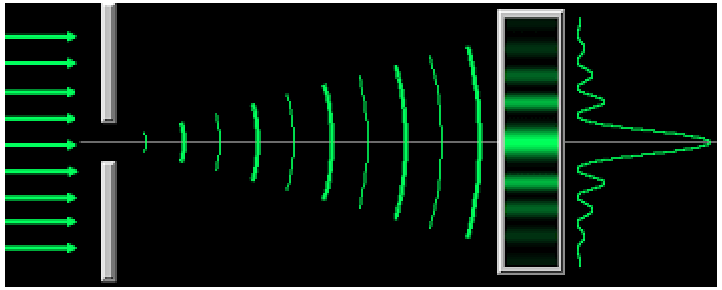
We finally reached the total current which is the function of the phase through the sample

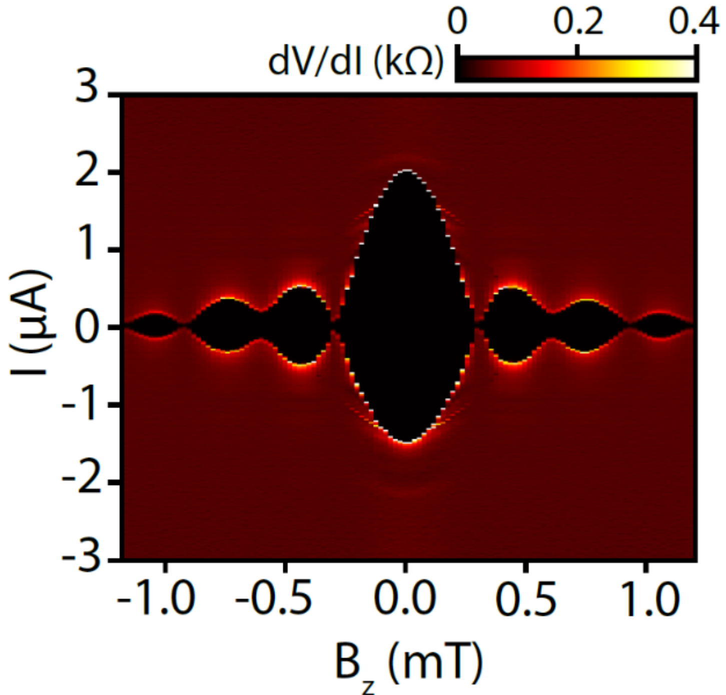
$$I_{max} = I_c \left| \frac{\sin(\pi\Phi/\Phi_0)}{\pi\Phi/\Phi_0} \right|$$

$$\Phi = HW(\lambda_1 + \lambda_2 + d)$$



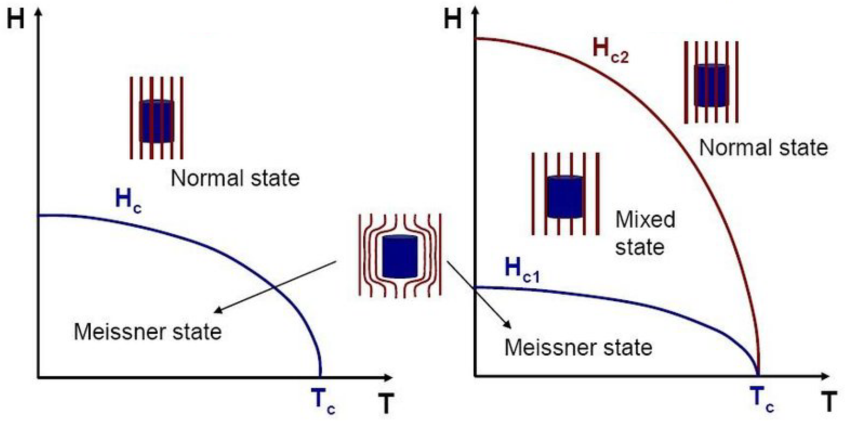
Experimentally, we can measure such a behavior that is similar to the single-slit diffraction pattern.





Quiz

1. Which is type I superconductor and which is type II

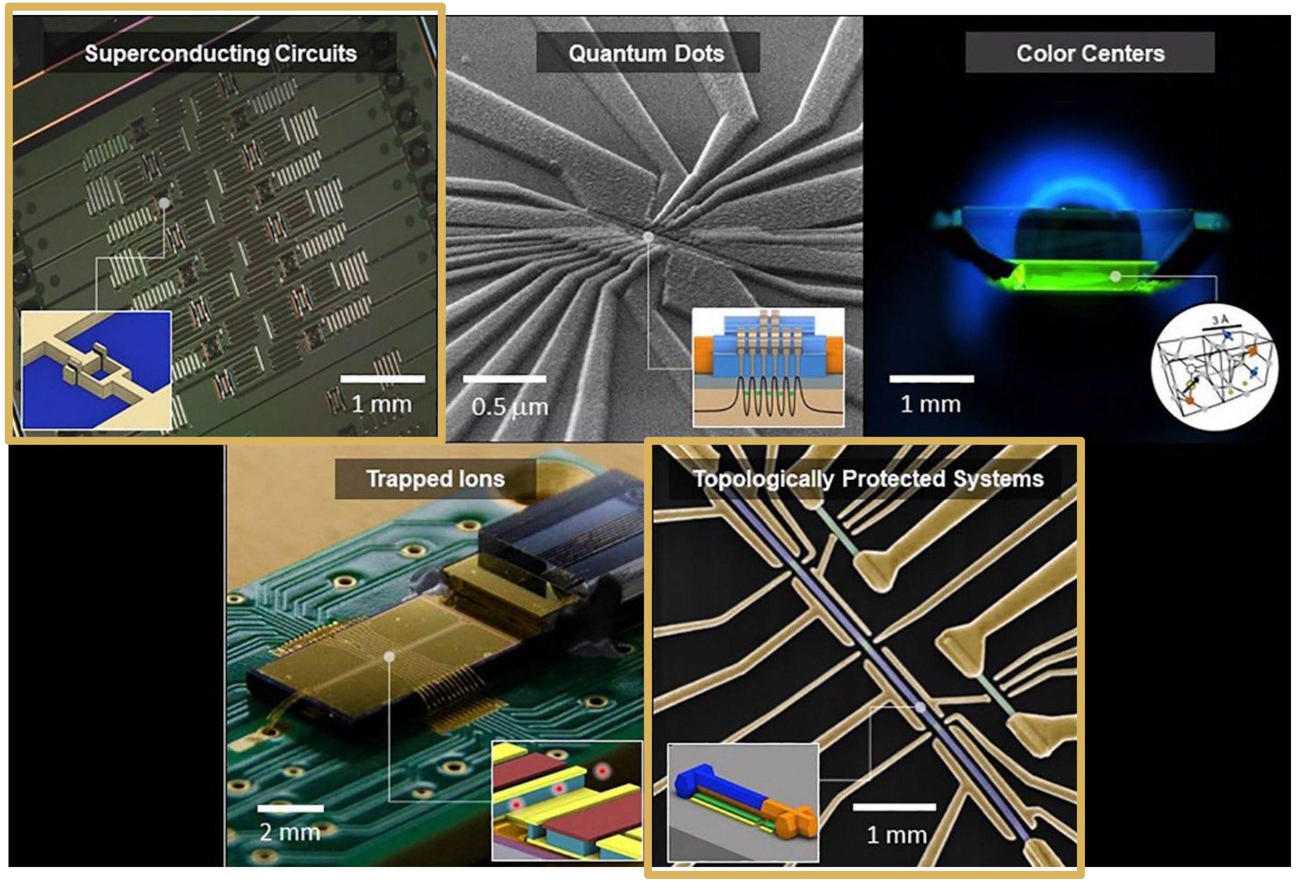


2. draw a diagram to show the DC Josephson effect, I vs. phi

$$I_s = I_c \sin(\phi)$$

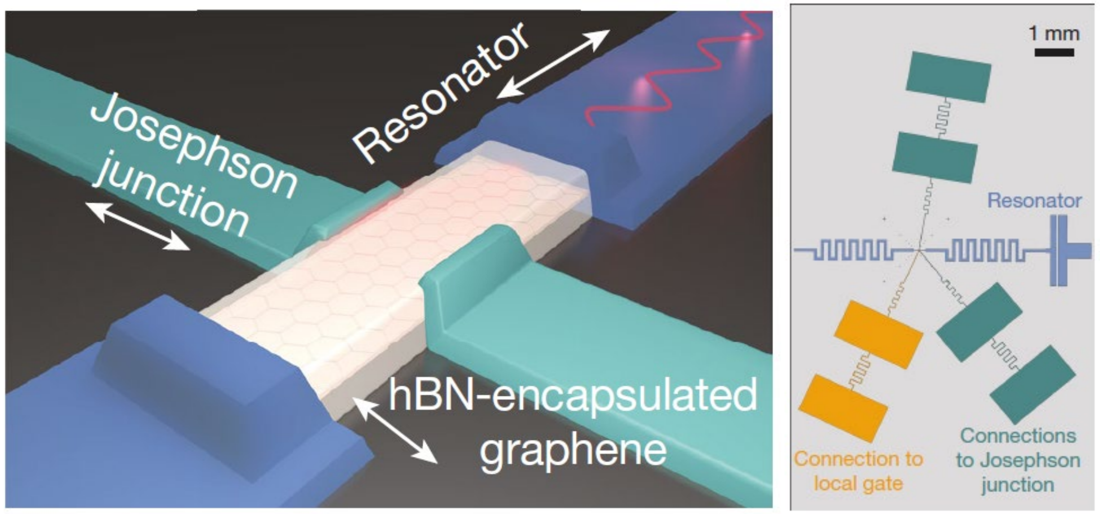
Application of Josephson junction

Develop Qubit



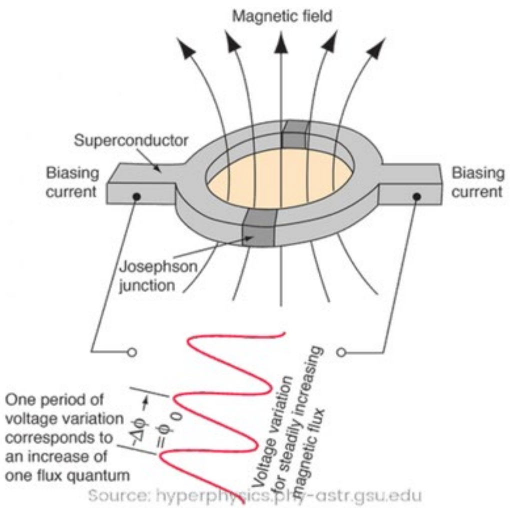
Source: Microsoft

Microwave Bolometer



G. Lee et al Nature (2020)

SQUID



Source: Quantum design

Here we focus on the applications for qubits.