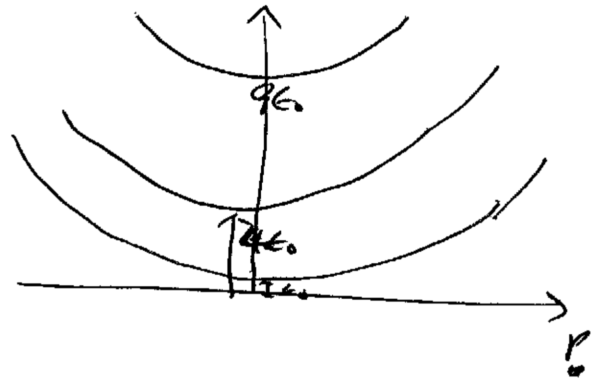
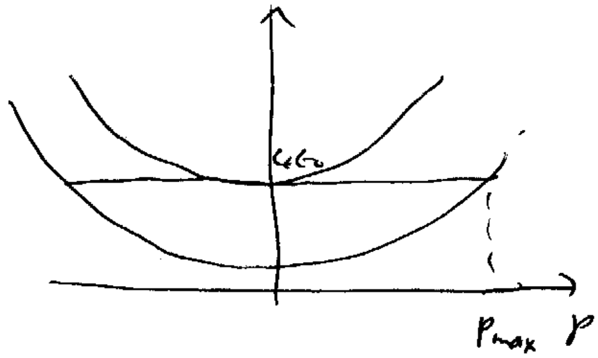
III.

(a) $$E(n,k) = n^2 \underbrace{\left(\frac{\pi^2\hbar^2}{2mw^2}\right)}_{\epsilon_0} + \frac{p_x^2+p_y^2}{2m}$$



(b) If only first subband occupied, maximum allowed energy is $4\epsilon_0$



At this point, the first subband is occupied to maximum momentum p_{max} such that

$$\epsilon_0 + \frac{p_{max}^2}{2m} = 4\epsilon_0$$

$$\therefore \quad p_{max}^2 = 2m(3\epsilon_0) = 2m\cdot 3\cdot\frac{\pi^2\hbar^2}{2mw^2} = \frac{3\pi^2\hbar^2}{w^2}$$

corresponding density of electrons per unit area

$$= \underset{\text{spin}}{\underset{\uparrow}{2}}\ \frac{\pi p_{max}^2}{(2\pi\hbar)^2} = \frac{p_{max}^2}{2\pi\hbar^2} = \frac{3\pi}{2w^2}$$

{ all states within a circle with radius p_{max} occupied }

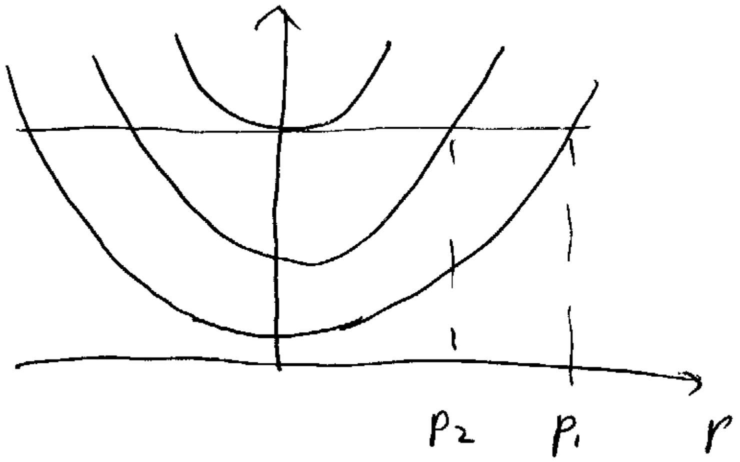
(c) If only first two subbands occupied,

maximum allowed energy $= 9\epsilon_0$

At this point,

first subband occupied to p_1

2nd subband to p_2



$$\epsilon_0 + \frac{p_1^2}{2m} = 9\epsilon_0$$

$$4\epsilon_0 + \frac{p_2^2}{2m} = 9\epsilon_0$$

$$p_1^2 = (2m)(8\epsilon_0) = (2m)\cdot 8 \cdot \frac{\pi^2\hbar^2}{2mW^2} = \frac{8\pi^2\hbar^2}{W^2}$$

$$p_2^2 = (2m)(5\epsilon_0) = \qquad = \frac{5\pi^2\hbar^2}{W^2}$$

total density per unit area

$$= 2\cdot\frac{(\pi p_1^2 + \pi p_2^2)}{(2\pi\hbar)^2} = \frac{(p_1^2+p_2^2)}{2\pi\hbar^2} = \frac{13\pi}{2W^2}$$

Remark:

Some of you recognized that the density of states per unit energy (including spin)* is $\frac{m}{\pi\hbar^2}$

Then, one can get answers to (b) & (c) respectively by:

(b): $\frac{m}{\pi\hbar^2}(4\epsilon_0 - \epsilon_0) = \frac{m}{\pi\hbar^2} \times 3 \times \frac{\pi^2\hbar^2}{2mw^2} = \frac{3\pi}{2w^2}$

(c) $\frac{m}{\pi\hbar^2}\Big[\underbrace{(9\epsilon_0-\epsilon_0)}_{\text{first band}} + \underbrace{(9\epsilon_0-4\epsilon_0)}_{\text{2nd band}}\Big] = \frac{m}{\pi\hbar^2}13\frac{\pi^2\hbar^2}{2mw^2}$

$= \frac{13\pi}{2w^2}$

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* see textbook, p. 234-237, 261

derivation: number of states per unit area in energy interval dE is given by

$$\underset{\substack{\uparrow\\ \text{spin}}}{2} \times \frac{2\pi k\,dk}{(2\pi)^2 dE} \underset{\substack{\uparrow\\ E=\frac{\hbar^2k^2}{2m}}}{=} \frac{m}{\pi\hbar^2}$$